Stochastic dynamical systems with weak contractivity property. The reflected affine stochastic recursion

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STOCHASTIC DYNAMICAL SYSTEMS WITH WEAK CONTRACTIVITY PROPERTIES. THE REFLECTED AFFINE STOCHASTIC RECURSION $X_n = |A_n X_{n-1} - B_n|$

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WITH A CHAPTER FEATURING RESULTS OF MARTIN BENDA

Abstract. Let (A_n, B_n) be a sequence of two-dimensional i.i.d. random variables with values in $\mathbb{R}^+ \times \mathbb{R}$. It induces the reflected affine stochastic recursion X_n^x which is given recursively by $X_0^x = x \geq 0$, and $X_{n+1}^x = |A_n X_n^x - B_{n+1}|$. When $A_n \equiv 1$, this is the classical reflected random walk on the half-line, and without the absolute value, it becomes an iteration of random affine transformations, studied by various authors in the last decades. In this paper, we study existence and uniqueness of invariant measures, as well as recurrence of this process. An initial chapter displays the unpublished work of Martin Benda on local contractivity, which merit publicity and provide an important tool for studying stochastic iterations.

1. Introduction

We start by reviewing two well known models.

First, let $(B_n)_{n\geq 0}$ be a sequence of i.i.d. real valued random variables. Then *reflected random walk* starting at $x \geq 0$ is the stochastic dynamical system given recursively by $X_0^x = x$ and $X_{n+1}^x = |X_n^x - B_{n+1}|$. The absolute value becomes meaningful when B_n assumes positive values with positive probability; otherwise we get an ordinary random walk on $\mathbb{R}$. Reflected random walk was described and studied by Feller [15]; apparently, it was first considered by von Schelling [30] in the context of telephone networks. In the case when $B_n \geq 0$, Feller [15] and Knight [22] have computed an invariant measure for the process when the Y_n are non-lattice random variables, while Boudiba [6], [7] has provided such a measure when the Y_n are lattice variables. Leguesdron [23], Boudiba [7] and Benda [3] have also studied its uniqueness (up to constant factors). When that invariant measure has finite total mass – which holds if and only if $\mathsf{E}(B_1) < \infty$ – the process is (topologically) recurrent: with probability 1, it returns infinitely often to each open set that is charged by the invariant measure. Indeed, it is positive recurrent in the sense that the mean return time is finite. More general recurrence criteria were provided by Smirnov [31] and Rabeherimanana [28], and also in our unpublished paper [27]: basically, recurrence holds when $\mathsf{E}\big(\sqrt{B_1}\big)$ or quantities of more or less the same order are finite. In the present paper, we shall briefly touch the situation when the B_n are not necessarily positive.

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Second, let $(A_n, B_n)_{n\ge 0}$ be a sequence of i.i.d. random variables in $\mathbb{R}^+ \times \mathbb{R}$, where $\mathbb{R}^+ = (0\,,\,\infty)$. The associated *affine stochastic recursion* on $\mathbb{R}$ is given by $Y_0^x = x \in \mathbb{R}$ and $Y_{n+1}^x = A_n Y_n^x + B_{n+1}$. There is an ample literature on this process, which can be interpreted in terms of a random walk on the affine group. That is, one applies products of affine matrices:

$$\begin{pmatrix} Y_n^x \\ 1 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{n-1} & B_{n-1} \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Products of affine transformations were one of the first examples of random walks on non-commutative groups, see Grenander [18]. Among the large body of further work, we mention Grincevičjus [19], [20], Elie [12], [13], [14]. and in particular the papers by Babillot, Bougerol and Elie [2] and Brofferio [8].

As an application of the results of the present paper, we study the synthesis of the above two processes. This is the variant of the affine recursion which is forced to stay non-negative: whenever it reaches the negative half-axis, its sign is changed. Thus, we have i.i.d. random variables $(A_n, B_n)_{n\ge 0}$ in $\mathbb{R}^+ \times \mathbb{R}$, and our process is

$$X_0^x = x \ge 0 \quad \text{and} \quad X_n^x = |A_n X_n^x - B_{n+1}|\,. \tag{1.1}$$

We choose the minus sign in the recursion in order to underline the analogy with reflected random walk. Indeed, the most typical situation is the one when $B_n \ge 0$. When $A_n \equiv 1$ then we are back at reflected random walk.

A simple family of key examples is given as follows.

(1.2) Example. Let $a > 1$ and let

$$A_n = \begin{cases} a & \text{with probability } \ p\,, \\ 1/a & \text{with probability } \ q = 1-p\,, \end{cases} \qquad B_n = 1 \ \text{ always.}$$

Thus, we randomly iterate the transformations $T_a = |ax-1|$ and $T_{1/a} = |x/a - 1|$, at each step choosing T_a or $T_{1/a}$ with probability p, resp. q. When $1 < a \le 2$, both transformations map the interval $[0\,,\,1]$ into itself, so that the process is best considered on that interval. When $a > 2$, it evolves on $[0\,,\infty)$.

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As often in this field, ideas that were first developped by Furstenberg, e.g. [17], play an important role at least in the background. Also, our methods are based in part on interesting and useful work of Benda in his PhD thesis [3] (in German) and the two subsequent preprints [4], [5] which were accepted for publication but have remained unpublished. Since we think that this material deserves to be documented in a publication, we include – with the consent of M. Benda whom we managed to contact – the next section on weak contractivity. The proofs that we give are "streamlined", and new aspects and results are added.

2. Local contractivity, based on the work of Benda

In this paper, stochastic dynamical systems are considered in the following setting. Let (X, d) be a proper metric space (i.e., closed balls are compact), and let $\mathfrak{G}$ be the monoid of all continuous mappings $\mathsf{X} \to \mathsf{X}$. It carries the topology of uniform convergence on compact sets. Now let $\widetilde{\mu}$ be a regular probability measure on $\mathfrak{G}$, and let $(F_n)_{n\geq 1}$ be a sequence of i.i.d. $\mathfrak{G}$-valued random variables (functions) with common distribution $\widetilde{\mu}$, defined on a suitable probability space $(\Omega, \mathfrak{A}, \mathsf{Pr})$. The measure $\widetilde{\mu}$ gives rise to the stochastic dynamical system (SDS) $\omega \mapsto X_n^x(\omega)$ defined by

$$X_0^x = x \in \mathsf{X}\,, \quad \text{and} \quad X_n^x = F_n(X_{n-1}^x)\,, \quad n \geq 1\,. \tag{2.1}$$

In the setting of our reflected affine recursion (1.1), we have $\mathsf{X} = [0\,,\,\infty)$ with the standard distance, and $F_n(x) = |A_n x - B_n|$, so that the measure $\widetilde{\mu}$ is the image of the distribution μ of the two-dimensional i.i.d. random variables (A_n, B_n) under the mapping $\mathbb{R} \times \mathbb{R}^+ \to \mathfrak{G}$, $(a, b) \mapsto f_{a,b}$, where $f_{a,b}(x) = |ax - b|$. Any SDS (2.1) is a Markov chain. The transition kernel is

$$P(x, U) = \mathsf{Pr}[X_1^x \in U] = \widetilde{\mu}(\{F \in \mathfrak{G} : F(x) \in U\})\,,$$

where U is a Borel set in X. The associated transition operator is given by

$$P\varphi(x) = \int_{\mathsf{X}} \varphi(y)\, dP(x, dy) = \mathsf{E}\big(\varphi(X_1^x)\big)\,,$$

where $\varphi : \mathsf{X} \to R$ is a measurable function for which this integral exists. The operator is Fellerian, that is, $P\varphi$ is continuous when φ is bounded and continuous.

We shall write $\mathcal{C}_c(\mathsf{X})$ for the space of compactly supported continuous functions $\mathsf{X} \to \mathbb{R}$.

(2.2) Definition. *(i)* The SDS is called *strongly contractive,* if for every $x \in \mathsf{X}$,

$$\mathsf{Pr}[d(X_n^x, X_n^y) \to 0 \quad \text{for all } y \in \mathsf{X}] = 1\,.$$

(ii) The SDS is called *locally contractive,* if for every $x \in \mathsf{X}$ and every compact $K \subset \mathsf{X}$,

$$\mathsf{Pr}[d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^x) \to 0 \quad \text{for all } y \in \mathsf{X}] = 1\,.$$

The notion of local contractivity was first introduced by Babillot, Bougerol and Elie [2] and was later exploited systematically by Benda, who (in personal comunication) also gives credit to unpublished work of his late PhD advisor Kellerer, compare with the posthumous publication [21].

Let $B(r)$ and $\overline{\mathsf{B}}(r)$, $r \in \mathbb{N}$, be the open and closed balls in X with radius r and fixed center $o \in \mathsf{X}$, respectively. $\overline{\mathsf{B}}(r)$ is compact by properness of X.

Using Kolomogorov's 0-1 law (and properness of X), one gets the following alternative.

(2.3) Lemma. *For a locally contractive SDS,*

$$\begin{array}{lll} \textit{either} & \mathsf{Pr}[d(X_n^x, x) \to \infty] = 0 & \textit{for all } x \in \mathsf{X}\,, \\ \textit{or} & \mathsf{Pr}[d(X_n^x, x) \to \infty] = 1 & \textit{for all } x \in \mathsf{X}\,. \end{array}$$

Proof. Consider

$$X_{m,m}^x = x \quad \text{and} \quad X_{m,n}^x = F_n \circ F_{n-1} \circ \ldots \circ F_{m+1}(x) \quad \text{for } m > n, \tag{2.4}$$

so that $X_n^x = X_{0,n}^x$. Then local contractivity implies that for each $x \in \mathsf{X}$, we have $\mathsf{Pr}(\Omega_0) = 1$ for the event Ω_0 consisting of all $\omega \in \Omega$ with

$$\lim_{n\to\infty} \mathbf{1}_{\mathsf{B}(r)}\big(X_{m,n}^x(\omega)\big) \cdot d\big(X_{m,n}^x(\omega), X_{m,n}^y(\omega)\big) = 0 \quad \text{for each } r \in \mathbb{N}\,,\ m \in \mathbb{N}_0\,,\ y \in \mathsf{X}. \tag{2.5}$$

Clearly, Ω_0 is invariant with respect to the shift of the sequence (F_n).

Now let $\omega \in \Omega_0$ be such that the sequence $\bigl(X_n^x(\omega)\bigr)_{n\ge 0}$ accumulates at some $w \in \mathsf{X}$. Fix m and set $v = X_m^x(\omega)$. Then also $\bigl(X_{m,n}^v(\omega)\bigr)_{n\ge m}$ accumulates at w. Now let $y \in \mathsf{X}$ be arbitrary. Then there is r such that $v, w, y \in \mathsf{B}(r)$. Therefore also $\bigl(X_{m,n}^y(\omega)\bigr)_{n\ge m}$ accumulates at w. In particular, the fact that $\bigl(X_n^x(\omega)\bigr)_{n\ge 0}$ accumulates at some point does not depend on the initial trajectory, i.e., on the specific realization of $F_1, \dots, F_m$. We infer that the set

$$\bigl\{\omega \in \Omega_0 : \bigl(X_n^x(\omega)\bigr)_{n\ge 0} \text{ accumulates in } \mathsf{X}\bigr\}$$

is a tail event of $(F_n)_{n\ge1}$. On its complement in Ω_0, we have $d(X_n^x, x) \to \infty$. □

If $d(X_n^x, x) \to \infty$ almost surely, then we call the SDS *transient.*

For $\omega \in \Omega$, let $\mathsf{L}^x(\omega)$ be the set of accumulation points of $\bigl(X_n^x(\omega)\bigr)$ in X. The following proof is much simpler than the one in [4].

(2.6) Lemma. *For any non-transient, locally contractive SDS, there is a set* $\mathsf{L} \subset \mathsf{X}$ *– the* attractor *or* limit set *– such that*

$$\mathsf{Pr}[\mathsf{L}^x(\cdot) = \mathsf{L} \ \textit{ for all } \ x \in \mathsf{C}] = 1\,,$$

Proof. The argument of the proof of Lemma 2.3 also shows the following. For every open $U \subset \mathsf{X}$,

$$\mathsf{Pr}[X_n^x \text{ accumulates in } U \text{ for all } \ x \in \mathsf{X}] \in \{0,1\}\,.$$

X being proper, we can find a countable basis $\{U_k : k \in \mathbb{N}\}$ of the topology of X, where each U_k is an open ball. Let $\mathbb{K} \subset \mathbb{N}$ be the (deterministic) set of all k such that the above probability is 1 for $U = U_k$. Then there is $\Omega_0 \subset \Omega$ such that $\mathsf{Pr}(\Omega_0) = 1$, and for every $\omega \in \Omega_0$, the sequence $\bigl(X_n^x(\omega)\bigr)_{n\ge 0}$ accumulates in U_k for some and equivalently all x precisely when $k \in \mathbb{K}$. Now, if $\omega \in \Omega_0$, then $y \in \mathsf{L}^x(\omega)$ precisely when $k \in \mathbb{K}$ for every k with $U_k \ni y$. We see that $\mathsf{L}^x(\omega)$ is the same set for every $\omega \in \Omega_0$. □

Thus, (X_n^x) is *(topologically) recurrent* on L when $\mathsf{Pr}[d(X_n^x, x) \to \infty] = 0$, that is, every open set that intersects L is visited infinitely often with probability 1.

For a Radon measure ν on X, its transform under P is written as νP, that is, for any Borel set $U \subset \mathsf{X}$,

$$\nu P(U) = \int_{\mathsf{X}} P(x, U)\, d\nu(x)\,.$$

Recall that ν is called *excessive,* when $\nu P \le \nu$, and *invariant,* when $\nu P = \nu$.

For two transition kernels P, Q, their product is defined as

$$PQ(x, U) = \int_{\mathsf{X}} Q(y, U)\, P(x, dy)\,.$$

In particular, P^k is the k-fold iterate The first part of the following is well-known; we outline the proof because it is needed in the second part, regarding $\mathsf{supp}(\nu)$.

(2.7) Lemma. *If the locally contractive SDS is recurrent, then every excessive measure* ν *is invariant. Furthermore,* $\mathsf{supp}(\nu) = \mathsf{L}$.

Proof. For any pair of Borel sets $U, V \subset X$, define the transition kernel $P_{U,V}$ and the measure ν_U by

$$P_{U,V}(x,B) = \mathbf{1}_U(x)\, P(x, B\cap V) \quad \text{and} \quad \nu_U(B) = \nu(U\cap B)\,,$$

where $B \subset \mathsf{X}$ is a Borel set. We abbreviate $P_{U,U} = P_U$. Also, consider the stopping time $\tau_x^U = \inf\{n \geq 1 : X_n^x \in U\}$, and for $x \in U$ let

$$P^U(x,B) = \mathsf{Pr}[\tau_x^U < \infty\,,\ X^x_{\tau_x^U} \in B]$$

be the probability that the first return of X_n^x to the set U occurs in a point of $B \subset X$. Then we have

$$\nu_U \geq \nu_U\, P_U + \nu_{U^c}\, P_{U^c,U}\,,$$

and by a typical inductive ("balayage") argument,

$$\nu_U \geq \nu_U \left(P_U + \sum_{k=0}^{n-1} P_{U,U^c}\, P_{U^c}^k\, P_{U^c,U} \right) + \nu_{U^c}\, P_{U^c}^n\, P_{U^c,U}\,.$$

In the limit,

$$\nu_U \geq \nu_U \left(P_U + \sum_{k=0}^{\infty} P_{U,U^c}\, P_{U^c}^k\, P_{U^c,U} \right) = \nu_U\, P^U\,.$$

Now suppose that U is open and relatively compact, and $U \cap \mathsf{L} \neq \emptyset$. Then, by recurrence, for any $x \in U$, we have $\tau_x^U < \infty$ almost surely. This means that P^U is stochastic, that is, $P^U(x,U) = 1$. But then $\nu_U\, P^U(U) = \nu_U(U) = \nu(U)$. Therefore $\nu_U = \nu_U\, P^U$ as (finite) Radon measures on X. We now can set $U = \mathsf{B}(r)$ and let $r \to \infty$. Then monotone convergence implies $\nu = \nu\, P$, and P is invariant.

Let us next show that $\mathsf{supp}(\nu) \subset \mathsf{L}$.

Take an open, relatively compact set V such that $V \cap \mathsf{L} = \emptyset$.

Now choose r large enough such that $U = B(r)$ contains V and intersects L. Let $Q = P^U$. We know from the above that $\nu_U = \nu_U\, Q = \nu_U\, Q^n$. We get

$$\nu(V) = \nu_U(V) = \int_U Q^n(x,V)\, d\nu_U(x)\,.$$

Now $Q^n(x,V)$ is the probability that the SDS starting at x visits V at the instant when it returns to U for the n-th time. Since

$$\mathsf{Pr}[X_n^x \in V \text{ for infinitely many } n] = 0\,,$$

it is an easy exercise to show that $Q^n(x,V) \to 0$. Since the measure ν_U has finite total mass, we can use dominated convergence to see that $\int_U Q^n(x,V)\, d\nu_U(x) \to 0$ as $n \to \infty$.

We conclude that $\nu(V) = 0$, and $\mathsf{supp}(\nu) \subset \mathsf{L}$.

Since $\nu\, P = \nu$, we have $f\big(\mathsf{supp}(\nu)\big) \subset \mathsf{supp}(\nu)$ for every $f \in \mathsf{supp}(\widetilde{\mu})$, where (recall) $\widetilde{\mu}$ is the distribution of the random functions F_n in $\mathfrak{G}$. But then almost surely $X_n^x \in \mathsf{supp}(\nu)$ for all $x \in \mathsf{supp}(\nu)$ and all n, that is, $\mathsf{L}^x(\omega) \subset \mathsf{supp}(\nu)$ for Pr-almost every ω. Lemma 2.6 yields that $\mathsf{L} \subset \mathsf{supp}(\nu)$. $\square$

The following holds in more generality than just for recurrent locally contractive SDS.

(2.8) Proposition. *If the locally contractive SDS is recurrent, then it possesses an invariant measure ν.*

Proof. Fix $\psi \in \mathcal{C}_c^+(\mathsf{X})$ such that its support intersects L. Recurrence implies that

$$\sum_{k=1}^{\infty} P^k\psi(x) = \infty \quad \text{for every } x \in \mathsf{X}.$$

The statement now follows from a result of LIN [25, Thm. 5.1]. □

Thus we have an invariant Radon measure ν with $\nu P = \nu$ and $\mathsf{supp}(\nu) = \mathsf{L}$. It is now easy to see that the attractor depends only on $\mathsf{supp}(\widetilde{\mu}) \subset \mathfrak{G}$.

(2.9) Corollary. *In the recurrent case,* L *is the smallest non-empty closed subset of* X *with the property that* $f(\mathsf{L}) \subset \mathsf{L}$ *for every* $f \in \mathsf{supp}(\widetilde{\mu})$*.*

Proof. The reasoning at the end of the proof of Lemma 2.7 shows that L is indeed a closed set with that property. On the other hand, if $C \subset \mathsf{X}$ is closed, non-empty and such that $f(C) \subset C$ for all $f \in \mathsf{supp}(\widetilde{\mu})$ then $\big(X_n^x(\omega)\big)$ evolves almost surely within C when the starting point x is in C. But then $\mathsf{L}^x(\omega) \subset C$ almost surely, and on the other hand $\mathsf{L}^x(\omega) = \mathsf{L}$ almost surely. □

(2.10) Remark. Suppose that the SDS induced by the probability measure $\widetilde{\mu}$ on $\mathfrak{G}$ is not necessarily weakly contractive, resp. recurrent, but that there is another probability measure $\widetilde{\mu}'$ on $\mathfrak{G}$ which does induce a weakly contractive, recurrent SDS and which satisfies $\mathsf{supp}(\widetilde{\mu}) = \mathsf{supp}(\widetilde{\mu}')$. Let L be the limit set of this second SDS. Since it depends only on $\mathsf{supp}(\widetilde{\mu}')$, the results that we have so far yield that also for the SDS (X_n^x) associated with $\widetilde{\mu}$, L is the unique "essential class" in the following sense: it is the unique minimal non-empty closed subset of X such that

(1) for every open set $U \subset \mathsf{X}$ that intersects L and every starting point $x \in \mathsf{X}$, the sequence (X_n^x) visits U with positive probability, and

(2) if $x \in \mathsf{L}$ then $X_n^x \in \mathsf{L}$ for all n. □

For $\ell \geq 2$, we can lift each $f \in \mathfrak{G}$ to a continuous mapping

$$f^{(\ell)} : \mathsf{X}^\ell \to \mathsf{X}^\ell\,, \quad f^{(\ell)}(x_1, \dots, x_\ell) = \big(x_2, \dots, x_d, f(x_\ell)\big)\,.$$

In this way, the random mappings F_n induce the SDS $\big(F_n^{(\ell)} \circ \cdots \circ F_1^{(\ell)}(x_1, \dots, x_\ell)\big)_{n \geq 0}$ on $\mathsf{X}^{(\ell)}$. For $n \geq \ell - 1$ this is just $\big(X_{n-\ell+1}^{x_\ell}, \dots, X_n^{x_\ell}\big)\,.$

(2.11) Proposition. *If the SDS is locally contractive and recurrent on* X*, then so is the lifted process on* X^ℓ*. The limit set of the latter is*

$$\mathsf{L}^{(\ell)} = \big\{\big(x, f_1(x), f_2 \circ f_1(x), \dots, f_{\ell-1} \circ \dots f_1(x)\big) : x \in \mathsf{L},\ f_i \in \mathsf{supp}(\widetilde{\mu})\big\}^-\,,$$

and if the Radon measure ν *is invariant for the original SDS on* X*, then the measure* $\nu^{(\ell)}$ *is invariant for the lifted SDS on* X^ℓ*, where*

$$\int_{\mathsf{X}^\ell} f\, d\nu^{(\ell)} = \int_{\mathsf{X}} \cdots \int_{\mathsf{X}} f(x_1, \dots, x_\ell)\, P(x_{\ell-1}, dx_\ell)\, P(x_{\ell-2}, dx_{\ell-1}) \cdots P(x_1, dx_2)\, d\nu(x_1)\,.$$

Proof. It is a straightforward exercise to verify that the lifted SDS is locally contractive and has $\nu^{(\ell)}$ as an invariant measure. The main point is to prove that it is recurrent. For this purpose, we have to show that there is some relatively compact subset of X^ℓ that is visited infinitely often with positive probability. We start with a relatively compact open subset U_1 of X that intersects L. Then there are relatively compact sets $U_2, \dots, U_\ell$ such that

$$\begin{aligned}&\mathsf{Pr}[(F_1, \dots, F_{\ell-1}) \in \mathfrak{E}] = \alpha > 0\,, \quad \text{where}\\ &\mathfrak{E} = \{(f_1, \dots, f_{\ell-1}) \in \mathfrak{G}^{\ell-1} : f_1(U_1) \subset U_2\,, \dots, f_{\ell-1}(U_{\ell-1}) \subset U_\ell\}\,.\end{aligned}$$

We know that for arbitrary starting point $x \in U_1$, the SDS (X_n^x) returns to U_1 infinitely often. Let τ_n be the n-th return time to U_1. It is almost surely finite for each n. Let $\mathfrak{A}_{\tau_n}$ be the σ-algebra of the process up to τ_n. Consider the events

$$W_n = [X^x_{\tau_{\ell n}+1} \in U_2\,,\ X^x_{\tau_{\ell n}+2} \in U_3\,, \dots,\ X^x_{\tau_{\ell n}+\ell-1} \in U_\ell].$$

Since $X_{\tau_n} \in U_1$, and since τ_n is a stopping time, we have

$$\mathsf{E}\Big(\mathsf{Pr}\big(W_n^c \mid \mathfrak{A}_{\tau_{\ell n}}\big)\Big) \le \mathsf{E}\Big(\mathsf{Pr}\big[(F_{\tau_{\ell n}+1}, \dots, F_{\tau_{\ell n}+\ell-1}) \notin \mathfrak{E} \mid \mathfrak{A}_{\tau_{\ell n}}\big]\Big) = 1 - \alpha\,.$$

Therefore, for $1 \le m < n$,

$$\begin{aligned}\mathsf{Pr}\left(\bigcap_{k=m}^{n} W_k^c\right) &= \mathsf{E}\left(\mathsf{Pr}\left(\underbrace{\bigcap_{k=m}^{n-1} W_k^c}_{\in\, \mathfrak{A}_{\tau_{\ell n}}} \cap W_n^c \;\middle|\; \mathfrak{A}_{\tau_{\ell n}}\right)\right)\\ &= \mathsf{Pr}\left(\bigcap_{k=m}^{n-1} W_k^c\right) \mathsf{E}\Big(\mathsf{Pr}\big(W_n^c \mid \mathfrak{A}_{\tau_{\ell n}}\big)\Big)\\ &\le (1-\alpha)\,\mathsf{Pr}\left(\bigcap_{k=m}^{n-1} W_k^c\right) \le \cdots \le (1-\alpha)^{n-m}\,.\end{aligned}$$

Letting $n \to \infty$ and taking the union over all m,

$$\mathsf{Pr}\left(\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} W_k^c\right) = 0\,.$$

Therefore $\mathsf{Pr}(\limsup_k W_k) = 1$, and on that event, there are infinitely many n such that $\big(X^{x_\ell}_{n-\ell+1}, \dots, X^{x_\ell}_n\big) \in U_1 \times \cdots \times U_\ell$. In view of Lemma 2.3, the lifted SDS on X^ℓ is recurrent.

Now that we know this, it is clear from Corollary 2.9 that its attractor is the set L^ℓ, as stated. □

In view of Lemma 2.9, the SDS evolves within L when started at some point of L, and every invariant measure is supported in that set. Thus, we can consider the random mappings F_n and their distribution $\widetilde{\mu}$ just on L in the place of the whole of X. Recall the probability space $(\Omega, \mathfrak{A}, \Pr)$ on which the $\mathfrak{G}$-valued random variables F_n are defined, so that (more precisely) $F_n(x) = F_n(\omega, x)$ for $x \in \mathsf{L}$, $\omega \in \Omega$.

We can realize our SDS starting at $x \in \mathsf{L}$ on the space

$$\big(\mathsf{L}^{(\infty)}, \mathfrak{B}(\mathsf{L}^{(\infty)}), \mathsf{Pr}_x\big),$$

where

$$\mathsf{L}^{(\infty)} = \{\mathbf{x} = (x_n)_{n\ge 0} : (x_0, \dots, x_{\ell-1}) \in \mathsf{L}^{(\ell)} \text{ for every } \ell \ge 1\}\,,$$

$\mathfrak{B}(\mathsf{L}^{(\infty)})$ is the trace on $\mathsf{L}^{(\infty)}$ of the product Borel σ-algebra on $\mathsf{L}^{\mathbb{N}_0}$, and Pr_x is the image of the measure Pr under the mapping

$$\Omega \to \mathsf{L}^{(\infty)}, \quad \omega \mapsto \big(X_n^x(\omega)\big)_{n\ge 0}.$$

If $U \subset \mathsf{L}^{(\ell)}$ then we write $U^{(\infty)} = \{\mathbf{x} \in \mathsf{L}^{(\infty)} : (x_0,\dots,x_{\ell-1}) \in U\}$. The sets of this form, where $\ell \ge 1$ and U is (relatively) open in $\mathsf{L}^{(\ell)}$, generate the σ-agebra $\mathfrak{B}(\mathsf{L}^{(\infty)})$. Given the invariant measure ν with support L, we next consider the measure on $\mathsf{L}^{(\infty)}$ defined by

$$\mathsf{Pr}_\nu = \int_{\mathsf{L}} \mathsf{Pr}_x \, d\nu(x).$$

It is a probability measure only when ν is a probability measure on L. In general, it is σ-finite and invariant with respect to the time shift $T : \mathsf{L}^{(\infty)} \to \mathsf{L}^{(\infty)}$.

(2.12) Lemma. *If the SDS is locally contractive and recurrent, then T is conservative.*

Proof. It is sufficient to verify that for every set of the form $U^{(\infty)}$, where U is relatively open in $\mathsf{L}^{(\ell)}$ with $\ell \ge 1$, one has

$$\mathsf{Pr}_\nu\Big(\Big\{\mathbf{x} = (x_n)_{n\ge 0} \in \mathsf{L}^{\mathbb{N}_0} : 0 < \sum_{n=0}^{\infty} \mathbf{1}_U^{(\infty)}(T^n\mathbf{x}) < \infty\Big\}\Big) = 0.$$

But

$$\mathsf{Pr}_x\Big(\Big\{\mathbf{x} : \sum_{n=0}^{\infty} \mathbf{1}_U(T^n\mathbf{x}) = \infty\Big\}\Big) = \mathsf{Pr}[(X_n^x,\dots,X_{n+\ell-1}^x) \in U \text{ infinitely often}] = 1,$$

by Proposition 2.11. □

The uniqueness part of the following theorem is contained in [3] and [4]; see also Brofferio [8, Thm. 3], who considers SDS of affine mappings. We re-elaborate the proof in order to be able to conclude that our SDS is ergodic with respect to T. (This, as well as Proposition 2.11, is new with respect to Benda's work.)

(2.13) Theorem. *For a recurrent locally contractive SDS, let ν be the measure of Proposition 2.8. Then the shift T on $\mathsf{L}^{(\infty)}$ is ergodic with respect to Pr_ν.*

In particular, ν is the unique invariant Radon measure for the SDS up to multiplication with constants.

Proof. Let $\mathfrak{I}$ be the σ-algebra of the T-invariant sets in $\mathfrak{B}(\mathsf{L}^{(\infty)})$. For $\varphi \in L^1(\mathsf{L}^{(\infty)}, \mathsf{Pr}_\nu)$, we write $\mathsf{E}_\nu(\varphi) = \int \varphi \, d\mathsf{Pr}_\nu$ and $\mathsf{E}_\nu(\varphi \mid \mathfrak{I})$ for the conditional "expectation" of φ with respect to $\mathfrak{I}$. The quotation marks refer to the fact that it does not have the meaning of an expectation when ν is not a probability measure. As a matter of fact, what is well defined in the latter case are quotients $\mathsf{E}_\nu(\varphi \mid \mathfrak{I})/\mathsf{E}_\nu(\psi \mid \mathfrak{I})$ fort suitable $\psi \ge 0$; compare with the explanations in Revuz [29, pp. 133–134].

In view of Lemma 2.12, we can apply the ergodic theorem of Chacon and Ornstein [9], see also [29, Thm.3.3]. Choosing an arbitrary function $\psi \in L^1(\mathsf{L}^{(\infty)}, \mathsf{Pr}_\nu)$ with

$$\mathsf{Pr}_\nu\Big(\Big\{\mathbf{x} \in \mathsf{L}^{(\infty)} : \sum_{n=0}^{\infty} \psi(T^n\mathbf{x}) < \infty\Big\}\Big) = 0, \tag{2.14}$$

one has for every $\varphi \in L^1(\mathsf{L}^{(\infty)}, \mathsf{Pr}_\nu)$

$$(2.15)\qquad \lim_{n\to\infty} \frac{\sum_{k=0}^n \varphi(T^k\mathbf{x})}{\sum_{k=0}^n \psi(T^k\mathbf{x})} = \frac{\mathsf{E}_\nu(\varphi \mid \mathfrak{I})}{\mathsf{E}_\nu(\psi \mid \mathfrak{I})} \quad \text{for } \mathsf{Pr}_\nu\text{-almost every } \mathbf{x} \in \mathsf{L}^{(\infty)}.$$

In order to show ergodicity of T, we need to show that the right hand side is just

$$\frac{\mathsf{E}_\nu(\varphi)}{\mathsf{E}_\nu(\psi)}\,.$$

It is sufficient to show this for non-negative functions that depend only on finitely many coordinates. For a function φ on $\mathsf{L}^{(\ell)}$, we also write φ for its extension to $\mathsf{L}^{(\infty)}$, given by $\varphi(\mathbf{x}) = \varphi(x_0, \dots, x_{\ell-1})$.

That is, we need to show that for every $\ell \geq 1$ and non-negative Borel functions φ, ψ on $\mathsf{L}^{(\ell)}$, with ψ satisfying (2.14),

$$(2.16)\qquad \begin{aligned} &\lim_{n\to\infty} \frac{\sum_{k=0}^n \varphi\big(X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega)\big)}{\sum_{k=0}^n \psi\big((X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega))\big)} = \frac{\int_{\mathsf{L}} \mathsf{E}\big(\varphi(X_0^y, \dots, X_{\ell-1}^y)\big)\, d\nu(y)}{\int_{\mathsf{L}} \mathsf{E}\big(\psi(X_0^y, \dots, X_{\ell-1}^y)\big)\, d\nu(y)} \\ &\text{for } \nu\text{-almost every } x \in \mathsf{L} \text{ and } \mathsf{Pr}\text{-almost every } \omega \in \Omega, \end{aligned}$$

when the integrals appearing in the right hand term are finite.

At this point, we observe that we need to prove (2.16) only for $\ell = 1$. Indeed, once we have the proof for this case, we can reconsider our SDS on $\mathsf{L}^{(\ell)}$, and using Propostion 2.11, our proof for $\ell = 1$ applies to the new SDS as well.

So now let $\ell = 1$. By regularity of ν, we may assume that φ and ψ are non-negative, compactly supported, continuous functions on L that both are non-zero.

We consider the random variables $S_n^x\varphi(\omega) = \sum_{k=0}^n \varphi\big(X_k^x(\omega)\big)$ and $S_n^x\psi(\omega)$. Since the SDS is recurrent, both functions satisfy (2.14), i.e., we have almost surely that $S_n^x\varphi$ and $S_n^x\psi > 0$ for all but finitely many n and all x. We shall show that

$$(2.17)\qquad \lim_{n\to\infty} \frac{S_n^x\varphi}{S_n^x\psi} = \frac{\int_{\mathsf{L}} \varphi\, d\nu}{\int_{\mathsf{L}} \psi\, d\nu} \quad \mathsf{Pr}\text{-almost surely and for } \textit{every } x \in \mathsf{L}\,,$$

which is more than what we need (namely that it just holds for ν-almost every x). We know from (2.15) that the limit exists in terms of conditional expectations for ν-almost every x, so that we only have to show that that it is $\mathsf{Pr} \otimes \nu$-almost everywhere constant.

Step 1. Independence of x. Let $K_0 \subset \mathsf{L}$ be compact such that the support of φ is contained in K_0. Define $K = \{x \in \mathsf{L} : d(x, K_0) \leq 1\}$. Given $\varepsilon > 0$, let $0 < \delta \leq 1$ be such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $d(x,y) < \delta$.

By (2.15), there is x such that the limits $\lim_n S_n^x\mathbf{1}_K / S_n^x\varphi$ and $Z_{\varphi,\psi} = \lim_n S_n^x\varphi / S_n^x\psi$ exist and are finite Pr-almost surely.

Local contractivity implies that for this specific x and each $y \in \mathsf{X}$, we have the following. Pr-almost surely, there is a random $N \in \mathbb{N}$ such that

$$|\varphi(X_k^x) - \varphi(X_k^y)| \leq \varepsilon \cdot \mathbf{1}_K(X_k^x)$$

Therefore, for every $\varepsilon > 0$ and $y \in \mathsf{X}$

$$\limsup_{n\to\infty} \frac{|S_n^x\varphi - S_n^y\varphi|}{S_n^x\varphi} \leq \varepsilon \cdot \lim_{n\to\infty} \frac{S_n^x\mathbf{1}_K}{S_n^x\varphi} \quad \mathsf{Pr}\text{-almost surely.}$$

This yields that for every $y \in \mathsf{L}$,

$$\lim_{n\to\infty} \frac{S_n^x\varphi - S_n^y\varphi}{S_n^x\varphi} = 0\,, \quad \text{that is,} \quad \lim_{n\to\infty} \frac{S_n^y\varphi}{S_n^x\varphi} = 1 \quad \mathsf{Pr}\text{-almost surely.}$$

The same applies to ψ in the place of φ. We get that for all y,

$$\frac{S_n^x\varphi}{S_n^x\psi} - \frac{S_n^y\varphi}{S_n^y\psi} = \frac{S_n^y\varphi}{S_n^y\psi}\left(\frac{S_n^x\varphi}{S_n^y\varphi}\frac{S_n^y\psi}{S_n^x\psi} - 1\right) \to 0 \quad \mathsf{Pr}\text{-almost surely.}$$

In other terms, for the positive random variable $Z_{\varphi,\psi}$ given above in terms of our x,

$$\lim_{n\to\infty} \frac{S_n^y\varphi}{S_n^y\psi} = Z_{\varphi,\psi} \quad \mathsf{Pr}\text{-almost surely, for every } y \in \mathsf{L}\,.$$

Step 2. $Z_{\varphi,\psi}$ is a.s. constant. Recall the random variables $X_{m,n}^x$ of (2.4) and set $S_{m,n}^x\varphi(\omega) = \sum_{k=m+1}^n \varphi\big(X_{m,k}^x(\omega)\big)$, $n > m$. Then Step 1 also yields that for our given x and each m,

$$\text{(2.18)} \qquad \lim_{n\to\infty} \frac{S_{m,n}^y\varphi}{S_{m,n}^y\psi} = \lim_{n\to\infty} \frac{S_{m,n}^x\varphi}{S_{m,n}^x\psi} \quad \mathsf{Pr}\text{-almost surely, for every } x \in \mathsf{L}\,.$$

Let $\Omega_0 \subset \Omega$ be the set on which the convergence in (2.18) holds for all m, and both $S_n^x\varphi$ and $S_n^x\psi \to \infty$ on Ω_0. We have $\mathsf{Pr}(\Omega_0) = 1$. For fixed $\omega \in \Omega_0$ and $m \in \mathbb{N}$, let $y = X_m^x(\omega)$. Then (because in the ratio limit we can omit the first $m+1$ terms of the sums)

$$Z_{\varphi,\psi}(\omega) = \lim_{n\to\infty} \frac{S_n^x\varphi(\omega)}{S_n^x\psi(\omega)} = \lim_{n\to\infty} \frac{S_{m,n}^y\varphi(\omega)}{S_{m,n}^y\psi(\omega)} = \lim_{n\to\infty} \frac{S_{m,n}^x\varphi(\omega)}{S_{m,n}^x\psi(\omega)}.$$

Thus, $Z_{\varphi,\psi}$ is independent of $F_1, \dots, F_m$, whence it is constant by Kolmogorov's 0-1 law. This completes the proof of ergodicity. It is immediate from (2.17) that ν is unique up to multiplication by constants. $\square$

(2.19) Corollary. *Let the locally contractive SDS (X_n^x) be recurrent with invariant Radon measure ν. For relatively compact, open $U \subset \mathsf{X}$ which intersects L, consider the probability measure m_U on X defined by $\mathsf{m}_U(B) = \nu(B \cap U)/\nu(U)$. Consider the SDS with initial distribution m_U, and let τ^U be its return time to U.*

(a) *If $\nu(\mathsf{L}) < \infty$ then the SDS is* positive recurrent*, that is,*

$$\mathsf{E}(\tau^U) = \nu(\mathsf{L})/\nu(U) < \infty\,.$$

(b) *If $\nu(\mathsf{L}) = \infty$ then the SDS is* null recurrent*, that is,*

$$\mathsf{E}(\tau^U) = \infty\,.$$

This follows from the well known formula of Kac, see e.g. Aaronson [1, 1.5.5., page 44].

(2.20) Lemma. *In the positive recurrent case, let the invariant measure be normalised such that $\nu(\mathsf{L}) = 1$. Then, for every starting point $x \in X$, the sequence (X_n^x) converges in law to ν.*

Proof. Let $\varphi : \mathsf{X} \to \mathbb{R}$ be continuous and compactly supported. Since φ is uniformly continuous, local contractivity yields for all $x, y \in X$ that $\varphi(X_n^x) - \varphi(X_n^y) \to 0$ almost surely. By dominated convergence, $\mathsf{E}\big(\varphi(X_n^x) - \varphi(X_n^y)\big) \to 0$. Thus,

$$P^n\varphi(x) - \int \varphi\, d\nu = \int \big(P^n\varphi(x) - P^n\varphi(y)\big)\, d\nu(y) = \int \mathsf{E}\big(\varphi(X_n^x) - \varphi(X_n^y)\big)\, d\nu(y) \to 0$$

□

We conclude by recalling a different result that goes back to [17], which is related with contractivity. It certainly stands at the origin of the considerations that lead (among other) to the concept of local contractivity.

(2.21) Proposition. **[Furstenberg's contraction principle.]** *Let $(F_n)_{n\ge1}$ be i.i.d. continuous random mappings $\mathsf{X} \to \mathsf{X}$, and define the right process*

$$R_n^x = F_1 \circ \cdots \circ F_n(x)\,.$$

If there is an X-valued random variable Z such that

$$\lim_{n\to\infty} R_n^x = Z \quad \textit{almost surely for every } \; x \in \mathsf{X}\,,$$

then the distribution ν of the limit Z is the unique invariant probability measure for the SDS $X_n^x = F_n \circ \cdots \circ F_1(x)$.

A proof can be found, e.g., in Letac [24]. REFERENCE TO SOME BOOK.

3. Basic example: affine stochastic recursions

Here we briefly review the main known results on the SDS on $\mathsf{X} = \mathbb{R}$ given by

$$Y_0^x = x\,, \quad Y_{n+1}^x = A_n Y_n^x + B_{n+1}\,, \tag{3.1}$$

where $(A_n, B_n)_{n\ge0}$ is a sequence of i.i.d. random variables in $\mathbb{R}^+ \times \mathbb{R}$. The following results are known.

(3.2) Proposition. *If $\mathsf{E}(\log^+ A_n) < \infty)$ and*

$$-\infty \le \mathsf{E}(\log A_n) < 0$$

then (Y_n^x) is strongly contractive on $\mathbb{R}$.

If in addition $\mathsf{E}(\log^+ |B_n|) < \infty$ then the affine SDS has a unique invariant probability measure ν, and is (positive) recurrent on $\mathsf{L} = \mathsf{supp}(\nu)$.

Proof (outline). This is *the* classical application of Furstenberg's contraction principle. One verifies that for the associated right process,

$$R_n^x \to Z = \sum_{n=1}^{\infty} A_1 \cdots A_{n-1} B_n$$

almost surely for every $x \in \mathbb{R}$. The series that defines Z is almost surely abolutely convergent by the assumptions on the two expectations. Recurrence is easily deduced via Lemma 2.3. Indeed, we cannot have $|Y_n^x| \to \infty$ almost surely, because then by dominated convergence $\nu(U) = \nu\, P^n(U) \to 0$ for every relatively compact set U. □

(Recall that in the present paper, "recurrence" always refers to topological recurrence.)

(3.3) Proposition. *Suppose that* $\mathsf{Pr}[A_n = 1] < 1$ *and* $\mathsf{Pr}[A_n x + B_n = x] < 1$ *for all* $x \in \mathbb{R}$ *(non-degeneracy). If* $\mathsf{E}(|\log A_n|) < \infty$ *and* $\mathsf{E}(\log^+ B_n) < \infty$*, and if*

$$\mathsf{E}(\log A_n) = 0$$

then (Y_n^x) *is locally contractive on* $\mathbb{R}$.

If in addition $\mathsf{E}(|\log A_n|^2) < \infty$ *and* $\mathsf{E}\big((\log^+ |B_n|)^{2+\varepsilon}\big) < \infty$ *for some* $\varepsilon > 0$ *then the affine SDS has a unique invariant Radon measure* ν *with infinite mass, and it is (null) recurrent on* $\mathsf{L} = \mathsf{supp}(\nu)$.

This goes back to [2], with a small gap that was later filled in [4]. With the moment conditions as stated here, a nice and complete "geometric" proof is given in [8]: it is shown that under the stated hypotheses,

$$A_1 \cdots A_n \cdot \mathbf{1}_K(Y_n) \to 0 \quad \text{almost surely}$$

for very compact set K. Recurrence was shown earlier in [13, Lemma 5.49].

(3.4) Proposition. *If* $\mathsf{E}(|\log A_n|) < \infty)$ *and* $\mathsf{E}(\log^+ B_n) < \infty$*, and if*

$$\mathsf{E}(\log A_n) > 0$$

then (Y_n^x) *is transient, that is,* $|Y_n^x| \to \infty$ *almost surely for every starting point* $x \in \mathbb{R}$.

A proof is given, e.g., by Elie [14].

4. Iteration of random contractions

Let us now consider a more specific class of SDS: within $\mathfrak{G}$, we consider the closed submonoid $\mathfrak{L}_1$ of all *contractions* of X, i.e., mappings $f : \mathsf{X} \to \mathsf{X}$ with Lipschitz constant $\mathfrak{l}(f) \le 1$. We suppose that the probability measure $\widetilde{\mu}$ that governs the SDS is supported by $\mathfrak{L}_1$, that is, each random function F_n of (2.1) satisfies $\mathfrak{l}(F_n) \le 1$. In this case, one does not need local contractivity in order to obtain Lemma 2.3; this follows directly from properness of X and the inequality

$$D_n(x,y) \le d(x,y)\,, \quad \text{where} \quad D_n(x,y) = d(X_n^x, X_n^y)\,.$$

When $\mathsf{Pr}[d(X_n^x, x) \to \infty] = 0$ for every x, we should in general only speak of non-transience, since we do not yet have an attractor on which the SDS is topologically recurrent. Let $\mathfrak{S}(\widetilde{\mu})$ be the closed sub-semigroup of $\mathfrak{L}_1$ generated by $\mathsf{supp}(\widetilde{\mu})$.

(4.1) Remark. For strong contractivity it is sufficient that $\mathsf{Pr}[D_n(x,y) \to 0] = 1$ pointwise for all $x, y \in \mathsf{X}$.

Indeed, by properness, X has a dense, countable subset Y. If $K \subset \mathsf{X}$ is compact and $\varepsilon > 0$ then there is a finite $W \subset Y$ such that $d(y, W) < \varepsilon$ for every $y \in K$. Therefore

$$\sup_{y \in K} D_n(x,y) \le \underbrace{\max_{w \in W} D_n(x,w)}_{\to 0 \text{ a.s.}} + \varepsilon\,,$$

since $D_n(x,y) \le D_n(x,w) + D_n(w,y) \le D_n(x,w) + d(w,y)$.

The following key result of [3] (whose statement and proof we have slightly strengthened here) is inspired by [22, Thm. 2.2], where reflected random walk is studied; see also [23].

(4.2) Theorem. *If the SDS of contractions is non-transient, then it is strongly contractive if and only if $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function.*

Proof. Keeping Remark 4.1 in mind, first assume that If $D_n(x,y) \to 0$ almost surely for all x, y.

We can apply all previous results on (local) contractivity, and the SDS has the non-empty attractor L. If $x_0 \in \mathsf{L}$, then with probability 1 there is a random subsequence (n_k) such that $X_{n_k}^x \to x_0$ for every $x \in \mathsf{X}$, and by the above, this convergence is uniform on compact sets. Thus, the constant mapping $x \mapsto x_0$ is in $\mathfrak{S}(\widetilde{\mu})$.

Conversely, assume that $\mathfrak{S}(\widetilde{\mu})$ contains a constant function. Since $D_{n+1}(x,y) \le D_n(x,y)$, the limit $D_\infty(x,y) = \lim_n D_n(x,y)$ exists and is between 0 and $d(x,y)$. We set $w(x,y) = \mathsf{E}\big(D_\infty(x,y)\big)$. First of all, we claim that

$$\lim_{m\to\infty} w(X_m^x, X_m^y) = D_\infty(x,y) \quad \text{almost surely.} \tag{4.3}$$

To see this, consider $X_{m,n}^x$ as in (2.4). Then $D_{m,\infty}(x,y) = \lim_n d(X_{m,n}^x, X_{m,n}^y)$ has the same distribution as $D_\infty(x,y)$, whence $\mathsf{E}\big(D_{m,\infty}(x,y)\big) = w(x,y)$. Therefore, we also have

$$\mathsf{E}\big(D_{m,\infty}(X_m^x, X_m^y) \mid F_1, \dots, F_m\big) = w(X_m^x, X_m^y)\,.$$

On the other hand, $D_{m,\infty}(X_m^x, X_m^y) = D_\infty(x,y)$, and the bounded martingale

$$\Big(\mathsf{E}\big(D_\infty(x,y) | F_1, \dots, F_m\big)\Big)_{m\ge 1}$$

converges almost surely to $D_\infty(x,y)$. The proposed statement (4.3) follows.

Now let $\varepsilon > 0$ be arbitrary, and fix $x, y \in X$. We have to show that the event $A = [D_\infty(x,y) \ge \varepsilon]$ has probability 0.

(i) By non-transience,

$$\Pr\left(\bigcup_{r\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{n\ge m} [X_n^x,\ X_n^y \in \mathsf{B}(r)]\right) = 1\,.$$

On A, we have $D_n(x,y) \ge \varepsilon$ for all n. Therefore we need to show that $\Pr(A_r) = 0$ for each $r \in \mathbb{N}$, where

$$A_r = \bigcap_{m\in\mathbb{N}} \bigcup_{n\ge m} [X_n^x,\ X_n^y \in \mathsf{B}(r)\,,\ D_n(x,y) \ge \varepsilon]\,.$$

(ii) By assumption, there is $x_0 \in X$ which can be approximated uniformly on compact sets by functions of the form $f_k \circ \ldots \circ f_1$, where $f_j \in \mathsf{supp}(\widetilde{\mu})$. Therefore, given r there is $k \in \mathbb{N}$ such that

$$\Pr(C_{k,r}) > 0\,, \quad \text{where} \quad C_{k,r} = \left[\sup_{u\in\mathsf{B}(r)} d(X_k^u, x_0) \le \varepsilon/4\right].$$

On $C_{k,r}$ we have $D_\infty(u,v) \le D_k(u,v) \le \varepsilon/2$ for all $u,v \in \mathsf{B}(r)$. Therefore, setting $\delta = \Pr(C_{k,r}) \cdot (\varepsilon/2)$, we have for all $u,v \in \mathsf{B}(r)$ with $d(u,v) \ge \varepsilon$ that

$$\begin{aligned} w(u,v) &= \mathsf{E}\big(\mathbf{1}_{C_{k,r}}\, D_\infty(u,v)\big) + \mathsf{E}\big(\mathbf{1}_{\mathsf{X}\setminus C_{k,r}}\, D_\infty(u,v)\big) \\ &\le \Pr(C_{k,r}) \cdot (\varepsilon/2) + \big(1 - \Pr(C_{k,r})\big) \cdot d(u,v) \le d(u,v) - \delta\,. \end{aligned}$$

We conclude that on A_r, there is a (random) sequence (n_ℓ) such that

$$w(X^x_{n_\ell}, X^y_{n_\ell}) \le D_{n_\ell}(x,y) - \delta\,.$$

Passing to the limit on both sides, we see that (4.3) is violated on A_r, since $\delta > 0$. Therefore $\Pr(A_r) = 0$ for each r. $\square$

(4.4) Corollary. *If the semigroup $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function, then the SDS is locally contractive.*

Proof. In the transient case, X^x_n can visit any compact K only finitely often, whence $d(X^x_n, X^y_n) \cdot \mathbf{1}_K(X^x_n) = 0$ for all but finitely many n. In the non-transient case, we even have strong contractivity by Proposition 4.2. $\square$

5. Some remarks on reflected random walk

As outlined in the introduction, the refleced random walk on $[0\,,\,\infty)$ induced by a sequence $(B_n)_{n\ge 0}$ of i.i.d. real valued random variables is given by

$$X^x_0 = x \ge 0\,, \quad X^x_{n+1} = |X^x_n - B_{n+1}|\,. \tag{5.1}$$

Let μ be the distribution of the B_n, a probability measure on $\mathbb{R}$. The transition probabilities of reflected random walk are

$$P(x,U) = \mu(\{y : |x-y| \in U\})\,,$$

where $U \subset [0\,,\,\infty)$ is a Borel set. When $B_n \le 0$ almost surely, then (X^x_n) is an ordinary random walk (resulting from a sum of i.i.d. random variables). We shall exclude this, and we shall always assume to be in the *non-lattice* situation. That is,

$$\mathsf{supp}(\mu) \cap (0\,,\,\infty) \ne \emptyset\,, \quad \text{and there is no } \kappa > 0 \text{ such that} \quad \mathsf{supp}(\mu) \subset \kappa\cdot\mathbb{Z}\,. \tag{5.2}$$

For the lattice case, see [27].

For $b \in \mathbb{R}$, consider $g_b \in \mathfrak{L}_1\big([0\,,\,\infty)\big)$ given by $g_b(x) = |x-b|$. Then our reflected random walk is the SDS on $[0\,,\,\infty)$ induced by the random continuous contractions $F_n = g_{B_n}$, $n \ge 1$. The law $\widetilde{\mu}$ of the F_n is the image of μ under the mapping $b \mapsto g_b$.

In [23, Prop. 3.2], it is shown that $\mathfrak{S}(\widetilde{\mu})$ contains the constant function $x \mapsto 0$. Note that this statement and its proof in [23] are completely deterministic, regarding topological properties of the set $\mathsf{supp}(\mu)$. In view of Theorem 4.2 and Corollary 4.4, we get the following.

(5.3) Proposition. *Under the assumptions* (5.2)*, reflected random walk on* $[0\,,\,\infty)$ *is locally contractive, and strongly contractive if it is recurrent.*

A. Non-negative B_n.

We first consider the case when $\mathsf{Pr}[B_n \ge 0] = 1$. Let

$$N = \sup \mathsf{supp}(\mu) \quad \text{and} \quad \mathsf{L} = \begin{cases} [0\,,\,N], & \text{if } N < \infty\,, \\ [0\,,\,\infty), & \text{if } N = \infty\,. \end{cases}$$

The distribution function of μ is

$$F_\mu(x) = \mathsf{Pr}[B_n \le x] = \mu\big([0\,,\,x]\big),\ x \ge 0\,.$$

We next subsume basic properties that are due to [15], [22] and [23]; they do not depend on recurrence.

(5.4) Lemma. *Suppose that* (5.2) *is verified and that* $\mathsf{supp}(\mu) \subset [0\,,\,\infty)$. *Then the following holds.*

(a) *The reflected random walk with any starting point is absorbed after finitely many steps by the interval* L.

(b) *It is topologically irreducible on* L, *that is, for every* $x \in \mathsf{L}$ *and open set* $U \subset \mathsf{L}$, *there is* n *such that* $P^n(x,U) = \mathsf{Pr}[X_n^x \in U] > 0$.

(c) *The measure* ν *on* L *given by*

$$\nu(dx) = \big(1 - F_\mu(x)\big)\,dx\,,$$

where dx *is Lebesgue measure, is an invariant measure for the transition kernel* P.

At this point Lemma 2.7 implies that in the recurrent case, the above set is indeed the attractor, and ν is the unique invariant measure up to multiplication with constants. We now want to understand when we have recurrence.

(5.5) Theorem. *Suppose that* (5.2) *is verified and that* $\mathsf{supp}(\mu) \subset [0\,,\,\infty)$. *Then each of the following conditions implies the next one and is sufficient for recurrence of the reflecting random walk on* L.

(i) $$\mathsf{E}(B_1) < \infty$$

(ii) $$\mathsf{E}\big(\sqrt{B_1}\,\big) < \infty$$

(iii) $$\int_{[0\,,\,\infty)} \big(1 - F_\mu(x)\big)^2\,dx < \infty$$

(iv) $$\lim_{y\to\infty} \big(1 - F_\mu(y)\big) \int_0^y \big(F_\mu(y) - F_\mu(x)\big)\,dx = 0$$

In particular, one has positive recurrence precisely when $\mathsf{E}(B_1) < \infty$.

The proof of (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) is a basic exercise. For condition (i), see [22]. The implication (ii) $\implies$ recurrence is due to [31], while the recurrence condition (iii) was proved by ourselves in [27]. However, we had not been aware of [31], as well as of [28], where it is proved that already (iv) implies recurrence on L. Since ν has finite total mass precisely when $\mathsf{E}(B_1) < \infty$, the statement on positive recurrence follows from

Corollary 2.19. In this case, also Lemma 2.20 applies and yields that X_n^x converges in law to $\frac{1}{\nu(\mathsf{L})}\nu$. This was already obtained by [22].

Note that the "margin" between conditions (ii), (iii) and (iv) is quite narrow.

B. General reflected random walk.

We now drop the restriction that the random variables B_n are non-negative. Thus, the "ordinary" random walk $S_n = B_1 + \cdots + B_n$ on $\mathbb{R}$ may visit the positive as well as the negative half-axis. Since we assume that μ is non-lattice, the closed group generated by $\mathsf{supp}(\mu)$ is $\mathbb{R}$.

We start with a simple observation ([5] has a more complicated proof).

(5.6) Lemma. *If μ is symmetric, then reflected random walk is (topologically) recurrent if and only if the random walk S_n is recurrent.*

Proof. If μ is symmetric, then also $|S_n|$ is a Markov chain. Indeed, for a Borel set $U \subset [0\,,\,\infty)$,

$$\begin{aligned}\mathsf{Pr}[\,|S_{n+1}| \in U \mid S_n = x] &= \mu(-x+U) + \mu(-x-U) - \mu(-x)\,\delta_0(U)\\ &= \mathsf{Pr}[\,|S_{n+1}| \in U \mid S_n = -x]\,,\end{aligned}$$

and we see that $|S_n|$ has the same transition probabilities as the reflected random walk governed by μ. $\square$

Recall the classical result that when $\mathsf{E}(|B_1|) < \infty$ and $\mathsf{E}(B_1) = 0$ then S_n is recurrent; see Chung and Fuchs [11].

(5.7) Corollary. *If μ is symmetric and has finite first moment then reflected random walk is recurrent.*

Let $B_n^+ = \max\{B_n, 0\}$ and $B_n^- = \max\{-B_n, 0\}$, so that $B_n = B_n^+ - B_n^-$. The following is well-known.

(5.8) Lemma. *If* (a) $\mathsf{E}(B_1^-) < \mathsf{E}(B_1^+) \le \infty$*, or if* (b) $0 < \mathsf{E}(B_1^-) = \mathsf{E}(B_1^+) < \infty$*, then* $\limsup S_n = \infty$ *almost surely, so that there are infinitely many reflections.*

In general, we should exclude that $S_n \to -\infty$, since in that case there are only finitely many reflections, and reflected random walk tends to $+\infty$ almost surely. In the sequel, we assume that $\limsup S_n = \infty$ almost surely. Then the (non-strictly) ascending *ladder epochs*

$$\mathbf{s}(0) = 0\,, \quad \mathbf{s}(k+1) = \inf\{n > \mathbf{s}(k) : S_n \ge S_{\mathbf{s}(k)}\}$$

are all almost surely finite, and the random variables $\mathbf{s}(k+1) - \mathbf{s}(k)$ are i.i.d. We can consider the *embedded random walk* $S_{\mathbf{s}(k)}$, $k \ge 0$, which tends to ∞ almost surely. Its increments $\overline{B}_k = S_{\mathbf{s}(k)} - S_{\mathbf{s}(k-1)}$, $k \ge 1$, are i.i.d. non-negative random variables with distribution denoted $\overline{\mu}$. Furthermore, if $\overline{X}_k^x$ denotes the reflected random walk associated with the sequence $(\overline{B}_k)$, while X_n^x is our original reflected random walk associated with (B_n), then

$$\overline{X}_k^x = X_{\mathbf{s}(k)}^x\,,$$

since no reflection can occur between times $\mathbf{s}(k)$ and $\mathbf{s}(k+1)$. When $\Pr[B_n < 0] > 0$, one clearly has $\sup \mathsf{supp}(\overline{\mu}) = +\infty$. Lemma 5.4 implies the following.

(5.9) Corollary. *Suppose that* (5.2) *is verified,* $\Pr[B_n < 0] > 0$ *and* $\limsup S_n = \infty$. *Then*

(a) *reflected random walk is topologically irreducible on* $\mathsf{L} = [0\,,\,\infty)$*, and*

(b) *the embedded reflected random walk* $\overline{X}_k^x$ *is recurrent if and only the original reflected random walk is recurrent.*

Proof. Statement (a) is clear.

Since both processes are locally contractive, each of the two processes is transient if and only if it tends to $+\infty$ almost surely: If $\lim_n X_n^x = \infty$ then clearly also $\lim_k X_{\mathbf{s}(k)}^x = \infty$ a.s. Conversely, suppose that $\lim_k \overline{X}_k^x \to \infty$ a.s. If $\mathbf{s}(k) \le n < \mathbf{s}(k+1)$ then $X_n^x \ge X_{\mathbf{s}(k)}^x$. (Here, k is random, depending on n and $\omega \in \Omega$, and when $n \to \infty$ then $k \to \infty$ a.s.) Therefore, also $\lim_n X_n^x = \infty$ a.s., so that (b) is also true. $\square$

We can now deduce the following.

(5.10) Theorem. *Suppose that* (5.2) *is verified and that* $\Pr[B_1 < 0] > 0$. *Then reflected random walk* (X_n^x) *is (topologically) recurrent on* $\mathsf{L} = [0\,,\,\infty)$*, if*

(a) $\mathsf{E}(B_1^-) < \mathsf{E}(B_1^+)$ *and* $\mathsf{E}\big(\sqrt{B_1^+}\,\big) < \infty$, *or if*

(b) $0 < \mathsf{E}(B_1^-) = \mathsf{E}(B_1^+)$ *and* $\mathsf{E}\Big(\sqrt{B_1^+}^{\,3}\Big) < \infty$.

Proof. We show that in each case the assumptions imply that $\mathsf{E}\big(\sqrt{\overline{B}_1}\big) < \infty$. Then we can apply Theorem 5.5 to deduce recurrence of $(\overline{X}_k^x)$. This in turn yields recurrence of (X_n^x) by Corollary 5.9.

(a) Under the first set of assumptions,

$$\begin{aligned}\mathsf{E}\Big(\sqrt{\overline{B}_1}\Big) &= \mathsf{E}\Big(\sqrt{B_1 + \ldots + B_{\mathbf{s}(1)}}\Big) \le \mathsf{E}\Big(\sqrt{B_1^+ + \ldots + B_{\mathbf{s}(1)}^+}\Big) \\ &\le \mathsf{E}\Big(\sqrt{B_1^+} + \ldots + \sqrt{B_{\mathbf{s}(1)}^+}\Big) = \mathsf{E}\Big(\sqrt{B_1^+}\Big) \cdot \mathsf{E}\big(\mathbf{s}(1)\big)\end{aligned}$$

by Wald's identity. Thus, we now are left with proving $\mathsf{E}\big(\mathbf{s}(1)\big) < \infty$. If $\mathsf{E}(B_1^+) < \infty$, then $\mathsf{E}(|B_1|) < \infty$ and $\mathsf{E}(B_1) > 0$ by assumption, and in this case it is well known that $\mathsf{E}\big(\mathbf{s}(1)\big) < \infty$; see e.g. [15, Thm. 2 in §XII.2, p. 396-397]. If $\mathsf{E}(B_1^+) = \infty$ then there is $M > 0$ such that $B_n^{(M)} = \min\{B_n\,, M\}$ (which has finite first moment) satisfies $\mathsf{E}(B_n^{(M)}) = \mathsf{E}(B_1^{(M)}) > 0$. The first increasing ladder epoch $\mathbf{s}^{(M)}(1)$ associated with $S_n^{(M)} = B_1^{(M)} + \ldots + B_n^{(M)}$ has finite expectation by what we just said, and $\mathbf{s}(1) \le \mathbf{s}^{(M)}(1)$. Thus, $\mathbf{s}(1)$ is integrable.

(b) If the B_n are centered, non-zero and $\mathsf{E}\big((B_1^+)^{1+a}\big) < \infty$, where $a > 0$, then $\mathsf{E}\big((\overline{B}_1)^a\big) < \infty$, as was shown by Chow and Lai [10]. In our case, $a = 1/2$. $\square$

We conclude our remarks on reflected random walk by discussing sharpness of the sufficient recurrence conditions $\mathsf{E}\left(\sqrt{B_1^+}^{\,3}\right) < \infty$ in the centered case, resp. $\mathsf{E}\left(\sqrt{B_1}\right) < \infty$ in the case when $B_1 \ge 0$.

(5.11) Example. Define a symmetric probability measure μ on $\mathbb{R}$ by

$$\mu(dx) = \frac{dx}{(1+|x|)^{1+a}}\,,$$

where $a > 0$ and c is the proper normalizing constant (and dx is Lebesgue measure). Then it is known that the associated symmetric random walk S_n on $\mathbb{R}$ is recurrent if and only if $a \ge 1$, see REFERENCE. By Lemma 5.6, the associated reflected random walk is also recurrent, but when $1 \le a \le 3/2$ then condition (b) of Theorem 5.10 does not hold.

Nevertheless, we can also show that in general, the sufficient condition $\mathsf{E}\left(\sqrt{\overline{B}_1}\right) < \infty$ for recurrence of reflected random walk with non-negative increments $\overline{B}_n$ is very close to being sharp. (We write $\overline{B}_n$ because we shall represent this as an embedded random walk in the next example.)

(5.12) Proposition. *Let μ_0 be a probability measure on $[0\,,\,\infty)$ which has a density $\phi_0(x)$ with respect to Lebesgue measure that is decreasing and satisfies*

$$\phi(x) \sim c\,(\log x)^b \big/ x^{3/2}\,, \quad \text{as } x \to \infty\,,$$

where $b > 1/2$ and $c > 0$. Then the associated reflected random walk on $[0\,,\,\infty)$ is transient.

Note that μ_0 has finite moment of order $\frac{1}{2} - \varepsilon$ for every $\varepsilon > 0$, while the moment of order $\frac{1}{2}$ is infinite.

The proof needs some preparation. Let (B_n) be i.i.d. random variables with values in $\mathbb{R}$ that have finite first moment and are non-constant and centered, and let μ be their common distribution.

The first *strictly ascending* and *strictly descending ladder epochs* of the random walk $S_n = B_1 + \ldots + B_n$ are

$$\mathbf{t}_+(1) = \inf\{n > 0 : S_n > 0\} \quad \text{and} \quad \mathbf{t}_-(1) = \inf\{n > 0 : S_n < 0\}\,,$$

respectively. They are almost surely finite. Let μ_+ be the distribution of $S_{\mathbf{t}_+(1)}$ and μ_- the distribution of $S_{\mathbf{t}_-(1)}$, and – as above – $\overline{\mu}$ the distribution of $\overline{B}_1 = S_{\mathbf{s}(1)}$. We denote the characteristic function associated with any probability measure σ on $\mathbb{R}$ by $\widehat{\sigma}(t)\,,\ t \in \mathbb{R}$. Then, following Feller [15, (3.11) in §XII.3], *Wiener-Hopf-factorization* tells us that

$$\mu = \overline{\mu} + \mu_- - \overline{\mu} * \mu_- \quad \text{and} \quad \overline{\mu} = u \cdot \delta_0 + (1-u)\cdot \mu_+\,,$$

$$\text{where} \quad u = \overline{\mu}(0) = \sum_{n=1}^{\infty} \mathsf{Pr}[S_1 < 0\,, \ldots, S_{n-1} < 0\,,\ S_n = 0] < 1\,.$$

Here $*$ is convolution. Note that when μ is absolutely continuous (i.e., absolutely continuous with respect to Lebesgue measure) then $u = 0$, so that

$$\overline{\mu} = \mu_+ \quad \text{and} \quad \mu = \mu_+ + \mu_- - \mu_+ * \mu_-\,. \tag{5.13}$$

(5.14) Lemma. *Let μ_0 be a probability measure on $[0\,,\,\infty)$ which has a decreasing density $\phi_0(x)$ with respect to Lebesgue measure. Then there is an absolutely continuous symmetric probability measure μ on $\mathbb{R}$ such that that the associated first (non-strictly) ascending ladder random variable has distribution μ_0.*

Proof. If μ_0 is the law of the first strictly ascending ladder random variable associated with some absolutely continuous, symmetric measure μ, then by (5.13) we must have $\mu_+ = \mu_0$ and $\mu_- = \check{\mu}_0$, the reflection of μ_0 at 0, and

$$\mu = \mu_0 + \check{\mu}_0 - \mu_0 * \check{\mu}_0\,. \tag{5.15}$$

We *define* μ in this way. The monotonicity assumption on μ_0 implies that μ is a probability measure: indeed, by the monotonicity assumption it is straightforward to check that the function $\phi = \phi_0 + \check{\phi}_0 - \phi_0 * \check{\phi}_0$ is non-negative; this is the density of μ.

The measure μ of (5.15) is non-degenerate and symmetric. If it induces a recurrent random walk (S_n), then the ascending and descending ladder epochs are a.s. finite. If (S_n) is transient, then $|S_n| \to \infty$ almost surely, but it cannot be $\mathsf{Pr}[S_n \to \infty] > 0$ since in that case this probaility had to be 1 by Kolmogorov's 0-1-law, while symmetry would yield $\mathsf{Pr}[S_n \to -\infty] = \mathsf{Pr}[S_n \to \infty] \le 1/2$. Therefore $\liminf S_n = -\infty$ and $\limsup S_n = +\infty$ almost surely, a well-known fact, see e.g. [15, Thm. 1 in §XII.2, p. 395]. Consequently, the ascending and descending ladder epochs are again a.s. finite. Therefore the probability measures μ_+ and $\mu_- = \check{\mu}_+$ (the laws of $S_{\mathbf{t}_\pm(1)}$) are well defined. By the uniqueness theorem of Wiener-Hopf-factorization [15, Thm. 1 in §XII.3, p. 401], it follows that $\mu_- = \check{\mu}_0$ and that the distribution of the first (non-strictly) ascending ladder random variable is $\overline{\mu} = \mu_0$. □

Proof of Proposition 5.12. Let μ be the symmetric measure associated with μ_0 according to (5.15) in Lemma 5.14. Then its characteristic function $\widehat{\mu}(t)$ is non-negative real. A well-known criterion says that the random walk S_n associated with μ is transient if and only if (the real part of) $1/\bigl(1 - \widehat{\mu}(t)\bigr)$ is integrable in a neighbourhood of 0. Returning to $\mu_0 = \mu_+$, it is a standard exercise (see [15, Ex. 12 in Ch. XVII, Section 12]) to show that there is $A \in \mathbb{C}$, $A \ne 0$ such that its characteristic function satisfies

$$\widehat{\mu_0}(t) = 1 + A\,\sqrt{t}\,(\log t)^b\,\bigl(1 + o(t)\bigr) \quad \text{as } t \to 0\,.$$

By (5.13),

$$1 - \widehat{\mu}(t) = \bigl(1 - \widehat{\mu_+}(t)\bigr)\bigl(1 - \widehat{\mu_-}(t)\bigr)\,.$$

We deduce

$$\widehat{\mu}(t) = 1 - |A|^2|t|(\log|t|)^{2b}\,\bigl(1 + o(t)\bigr) \quad \text{as } t \to 0\,.$$

The function $1/\bigl(1 - \widehat{\mu}(t)\bigr)$ is integrable near 0. By Lemma 5.6, the associated reflected random walk is transient. But then also the embedded reflected random walk associated with $S_{\mathbf{s}(n)}$ is transient by Corollary 5.9. This is the reflected random walk governed by μ_0. □

6. Stochastic dynamical systems induced by Lipschitz mappings

We now consider the situation when the i.i.d. random mappings $F_n : \mathsf{X} \to \mathsf{X}$ belong to the semigroup $\mathfrak{L} \subset \mathfrak{G}$ of Lipschitz mappings. Recall our notation $\mathfrak{l}(f)$ for the Lipschitz constant of $f \in \mathfrak{L}$. We assume that

$$\mathsf{Pr}[\mathfrak{l}(F_n) > 0] = 1\,, \quad \mathsf{Pr}[\mathfrak{l}(F_n) < 1] > 0\,, \quad \text{and} \quad \mathsf{Pr}[F_n(x) = x] < 1 \text{ for every } x \in \mathsf{X}\,. \tag{6.1}$$

In this situation, the real random variables

$$A_n = \mathfrak{l}(F_n) \quad \text{and} \quad |B|_n = d\big(F_n(o), o\big) \tag{6.2}$$

play an important role. Indeed, let (X_n^x) be the SDS starting at $x \in \mathsf{X}$ which is associated with the sequence (F_n), and for starting point $y \geq 0$, let (Y_n^y) the affine SDS on $[0\,,\infty)$ associated with $(A_n, |B|_n)$ according to (3.1). Then

$$d(X_n^x, o) \leq Y_n^{|x|}\,, \quad \text{where } \; |x| = d(x,o). \tag{6.3}$$

Thus, we can use the results of Section 3. First of all, Propositions 2.21, resp. 3.2 yield the following.

(6.4) Corollary. *Given the random i.i.d. Lipschitz mappings F_n, let A_n and $|B|_n$ be as in* (6.2).

If $\mathsf{E}(\log^+ A_n) < \infty$ and $-\infty \leq \mathsf{E}(\log A_n) < 0$ then the SDS (X_n^x) generated by the F_n is strongly contractive on X.

If in addition $\mathsf{E}(\log^+ |B|_n) < \infty$ then the SDS has a unique invariant probability measure ν on X, and it is (positive) recurrent on $\mathsf{L} = \mathsf{supp}(\nu)$.

Proof. Strong contractivity is obvious. When $\mathsf{E}(\log^+ |B|_n) < \infty$, (6.3) tells us that along with $(Y_n^{|x|})$ also (X_n^x) is positive recurrent. □

The interesting and much harder case is the one where $\log A_n$ is integrable and centered, that is, $\mathsf{E}(\log A_n) = 0$. The assumptions of Proposition 3.2, applied to A_n and $|B|_n$ of (6.2), will in general not imply that our SDS is locally contractive.

(6.5) Remark. In the centered case, we can apply Proposition 3.3 to $(Y_n^{|x|})$. Among its hypotheses, we still should check that $\mathsf{Pr}[A_n a + |B|_n = a] < 1$ for all $a \in \mathbb{R}$. When $a = 0$, this is the same as our assumption $\mathsf{Pr}\big(F_n(o) = o\big) < 1$. If $a \neq 0$ then observe that $A_n - 1$ assumes both positive and negative values with positive probability, so that the requirement is met. Therefore the affine SDS on $\mathbb{R}$ is locally contractive and recurrent on its limit set $\mathsf{L}_{\mathbb{R}}$, which is contained in $\mathbb{R}^+$ by construction. Note that it depends on the reference point $o \in \mathsf{X}$ through the definition of $|B|_n$.

In view of our assumptions (6.1), in the centered case we can modify the measure $\widetilde{\mu}$ on $\mathfrak{L}$ to obtain a new one, say $\widetilde{\mu}'$, which satisfies

$$\int_{\mathfrak{L}} \log \mathfrak{l}(f)\, d\widetilde{\mu}'(f) < 0\,.$$

For example, we may take $d\widetilde{\mu}'(f) = g(f)\, d\widetilde{\mu}(f)$, where $g(f) = c\cdot\big(\frac{1}{4}\mathbf{1}_{[\mathfrak{l}(f)\geq 1]}(f) + \frac{3}{4}\mathbf{1}_{[\mathfrak{l}(f)<1]}(f)\big)$ with the appropriate normalizing constant c. Then $\widetilde{\mu}'$ gives rise to a strongly contractive SDS. Let L be its limit set. Remark 2.10 tells us that also our original SDS governed by

$\widetilde{\mu}$ is topologically irreducible on L and that it evolves within L when started in a point of L. This set is given by Corollary 2.9. We may assume that the reference point o belongs to L.

In the sequel, we shall write

$$A_{m,m} = 1 \quad \text{and} \quad A_{m,n} = A_{m+1} \cdots A_{n-1} A_n \quad (n > m)\,.$$

(6.6) Theorem. *If in addition to* (6.1)*, one has*

$$\mathsf{E}(\log A_n) = 0\,, \quad \mathsf{E}(|\log A_n|^2) < \infty\,, \quad \textit{and} \quad \mathsf{E}\big((\log^+ |B_n|)^{2+\varepsilon}\big) < \infty \tag{6.7}$$

for some $\varepsilon > 0$, then the SDS is topologically recurrent on L. Moreover, for every $x \in \mathsf{X}$ (and not just $\in \mathsf{L}$) and every open set $U \subset \mathsf{X}$ that intersects L,

$$\mathsf{Pr}[X_n^x \in U \textit{ for infinitely many } n] = 1.$$

Proof. The (non-strictly) *descending* ladder epochs are

$$\boldsymbol{\ell}(0) = 0\,, \quad \boldsymbol{\ell}(k+1) = \inf\{n > \boldsymbol{\ell}(k) : A_{0,n} \le A_{0,\boldsymbol{\ell}(k)}\}$$

Since $(A_{0,n})$ is a recurrent multiplicative random walk on $\mathbb{R}^+$, these epochs are stopping times with i.i.d. increments. The induced SDS is $(\bar{X}_k^x)_{k\ge 0}$, where $\bar{X}_k^x = X_{\boldsymbol{\ell}(k)}^x$. It is also generated by random i.i.d. Lipschitz mappings, namely

$$\bar{F}_k = F_{\boldsymbol{\ell}(k)} \circ F_{\boldsymbol{\ell}(k)-1} \circ \cdots \circ F_{\boldsymbol{\ell}(k-1)+1}\,, \quad k \ge 1\,.$$

With the same stopping times, we also consider the induced affine recursion given by $\bar{Y}_k^{|x|} = Y_{\boldsymbol{\ell}(k)}^{|x|}$. It is generated by the i.i.d. pairs $(\bar{A}_k, \bar{B}_k)_{k\ge 1}$, where

$$\bar{A}_k = A_{\boldsymbol{\ell}(k-1),\,\boldsymbol{\ell}(k)} \quad \text{and} \quad \bar{B}_k = \sum_{j=\boldsymbol{\ell}(k-1)+1}^{\boldsymbol{\ell}(k)} |B|_j \, A_{j,\,\boldsymbol{\ell}(k)}\,.$$

It is known [13, Lemma 5.49] that under our assumptions, $\mathsf{E}(\log^+ \bar{A}_k) < \infty$, $\mathsf{E}(\log \bar{A}_k) < 0$ and $\mathsf{E}(\log^+ \bar{B}_k) < \infty$. Returning to $(\bar{X}_k^x)$, we have $\mathfrak{l}(\bar{F}_k) \le \bar{A}_k$ and $d\big(\bar{F}_k(o), o\big) \le \bar{B}_k$. Corollary 6.4 applies, and the induced SDS is strongly contractive. It has a unique invariant probability measure $\bar{\nu}$, and it is (positive) recurrent on $\bar{\mathsf{L}} = \mathsf{supp}(\bar{\nu})$. Moreover, for every starting point $x \in \mathsf{X}$ and each open set $U \subset \mathsf{X}$ that intersects $\bar{\mathsf{L}}$, we get that almost surely, $(\bar{X}_k^x)$ visits U infinitely often.

In view of the fact that the original SDS is topologically irreducible on L, we have $\bar{\mathsf{L}} \subset \mathsf{L}$. We now define a sequence of subsets of L by

$$\mathsf{L}_0 = \bar{\mathsf{L}} \quad \text{and} \quad \mathsf{L}_m = \bigcup\{f(\mathsf{L}_{m-1}) : f \in \mathsf{supp}(\widetilde{\mu})\}.$$

Then the closure of $\bigcup_m \mathsf{L}_m$ is a subset of L that is mapped into itself be every $f \in \mathsf{supp}(\widetilde{\mu})$. Corollary 2.9 yields that

$$\mathsf{L} = \big(\textstyle\bigcup_m \mathsf{L}_m\big)^-.$$

We now show by induction on m that for every starting point $x \in \mathsf{X}$ and every open set U that intersects L_m,

$$\mathsf{Pr}[X_n^x \in U \text{ for infinitely many } n] = 1,$$

and this will conclude the proof.

For $m = 0$, the statement is true. Suppose it is true for $m-1$. Given an open set U that intersects L_m, we can find an open, relatively compact set V that intersects L_{m-1} such that $\widetilde{\mu}(\{f \in \mathfrak{L} : f(V) \subset U\} = \alpha > 0$.

Given the starting point x, let $\bigl(\tau(n)\bigr)$ be the sequence of stopping times of the successive visits of (X_n^x) in V. By the induction hypothesis, all $\tau(n)$ are a.s. finite. The events $[X_{\tau(n)+1} \in U]$, $n \in \mathbb{N}$, are independent, and $\mathsf{Pr}[X_{\tau(n)+1} \in U] \ge \alpha$ for each n. By the reverse of the Borel-Cantelli lemma, we have almost surely that $X_{\tau(n)+1} \in U$ for infinitely many n. □

(6.8) Corollary. (a) *Under the assumptions* (6.1)*, every invariant Radon measure ν satisfies $\mathsf{L} \subset \mathsf{supp}(\nu)$.*

(b) *If in addition to* (6.1)*, one has* (6.7)*, then the SDS possesses an invariant Radon measure ν with $\mathsf{supp}(\nu) = \mathsf{L}$. Furthermore, the transition operator P is a conservative contraction of $L^1(\mathsf{X}, \nu)$ for every invariant measure ν.*

Proof. (a) Let ν be invariant. The argument at the end of the proof of Lemma 2.7 shows that $f\bigl(\mathsf{supp}(\nu)\bigr) \subset \mathsf{supp}(\nu)$ for all $f \in \mathsf{supp}(\widetilde{\mu})$. As explained above, Corollary 2.9 applies here and yields statement (a).

(b) Theorem 6.6 yields conservativity. Indeed, let $\mathsf{B}(r)$ be a ball that intersects L. For every starting point $x \in \mathsf{X}$, the SDS (X_n^x) visits $\mathsf{B}(r)$ infinitely often with probability 1. We can choose $\varphi \in \mathcal{C}_c^+(\mathsf{X})$ such that $\varphi \ge 1$ on $\mathsf{B}(r)$. Then

$$\sum_{k=1}^{\infty} P^k \varphi(x) = \infty \quad \text{for every } x \in \mathsf{X}, \tag{6.9}$$

The existence of an invariant Radon measure follows once more from [25, Thm. 5.1], and conservativity of P on $L^1(\mathsf{X}, \nu)$ follows, see e.g. [29, Thm. 5.3]. If right from the start we consider the whole process only on L with the induced metric, then we obtain an invariant measure ν with $\mathsf{supp}(\nu) = \mathsf{L}$. □

Note that unless we know that the SDS is locally contractive, we cannot argue right away that every invariant measure must be supported exactly by L. The assumptions (6.1) & (6.7) will in general not imply local contractivity, as we shall see below. Thus, the question of uniqueness of the invariant measure is more subtle. For a sufficient condition that requires a more restrictive (Harris type) notion of irreducibility, see [25, Def. 5.4 & Thm. 5.5].

Hyperbolic extension

In order to get closer to answering the uniqueness question in a more "topological" spirit, we also want to control the Lipschitz constants A_n. We shall need to distinguish between two cases.

A. Non-lattice case

If the random variables $\log A_n$ are non-lattice, i.e., there is no $\kappa > 0$ such that $\log A_n \in$

$\kappa \cdot \mathbb{Z}$ almost surely, then we consider consider the extended SDS

$$\widehat{X}_n^{x,a} = (X_n^x, A_n A_{n-1} \cdots A_1 a) \tag{6.10}$$

on extended space $\widehat{\mathsf{X}} = \mathsf{X} \times \mathbb{R}^+$, with initial point $(x,a) \in \widehat{\mathsf{X}}$.

B. Lattice case

Otherwise, there is a maximal $\kappa > 0$ such that $\log A_n \in \kappa \cdot \mathbb{Z}$ almost surely. Then we consider again the extended SDS (6.10), but now the extended space is $\widehat{\mathsf{X}} = \mathsf{X} \times \exp(\kappa \cdot \mathbb{Z})$, where of course $\exp(\kappa \cdot \mathbb{Z}) = \{e^{\kappa m} : m \in \mathbb{Z}\}$. The initial point (x,a) now has to be such that also $a \in \exp(\kappa\mathbb{Z})$.

Consider the hyperbolic upper half plane $\mathbb{H} \subset \mathbb{C}$ with the Poincaré metric

$$\theta(z,w) = \log \frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\,,$$

where $z, w \in \mathbb{H}$ and $\bar{w}$ is the complex conjugate of w. We use it to define a "hyperbolic" metric on $\widehat{X}$ by

$$\begin{aligned} \hat{d}\big((x,a),(y,b)\big) &= \theta\big(\mathfrak{i}\, a, d(x,y) + \mathfrak{i}\, b\big) \\ &= \log \frac{\sqrt{d(x,y)^2 + (a+b)^2} + \sqrt{d(x,y)^2 + (a-b)^2}}{\sqrt{d(x,y)^2 + (a+b)^2} - \sqrt{d(x,y)^2 + (a-b)^2}}\,. \end{aligned} \tag{6.11}$$

It is a good exercise, using the specific properties of θ, to verify that this is indeed a metric. The metric space $(\widehat{\mathsf{X}}, \hat{d})$ is again proper, and for any $a > 0$, the embedding $\mathsf{X} \to \widehat{\mathsf{X}}$, $x \mapsto (x,a)$, is a homeomorphism.

(6.12) Lemma. *Let $f : \mathsf{X} \to \mathsf{X}$ be a Lipschitz mapping with Lipschitz constant $\mathfrak{l}(f) > 0$. Then the mapping $\hat{f} : \widehat{\mathsf{X}} \to \widehat{\mathsf{X}}$, defined by*

$$\hat{f}(x,a) = \big(f(x), \mathfrak{l}(f)a\big)$$

is a contraction of $(\widehat{\mathsf{X}}, \hat{d})$ with Lipschitz constant 1.

Proof. We have by the dilation invariance of the hyperbolic metric

$$\begin{aligned} \tilde{d}\Big(\hat{f}(x,a), \hat{f}(y,b)\Big) &= \theta\Big(\mathfrak{i}\, \mathfrak{l}(f)a, d\big(f(x), f(y)\big) + \mathfrak{i}\, \mathfrak{l}(f)b\Big) \le \theta\Big(\mathfrak{i}\, \mathfrak{l}(f)a, \mathfrak{l}(f)d(x,y) + \mathfrak{i}\, \mathfrak{l}(f)b\Big) \\ &= \theta\Big(\mathfrak{i}\, a, d(x,y) + \mathfrak{i}\, b\Big) = \widehat{d}\Big((x,a),(y,b)\Big)\,. \end{aligned}$$

Thus, $\mathfrak{l}(\hat{f}) \le 1$. Furthermore, if $\varepsilon > 0$ and $x, y \in \mathsf{X}$ are such that $d\big(f(x), f(y)\big) \ge (1-\varepsilon)\mathfrak{l}(f)d(x,y)$ then the we obtain in the same way that

$$\hat{d}\Big(\tilde{f}(x,a), \tilde{f}(y,b)\Big) \ge \theta\Big(\mathfrak{i}\, a, (1-\varepsilon)d(x,y) + \mathfrak{i}\, b\Big)\,.$$

when $\varepsilon \to 0$, the right hand side tends to $\hat{d}\big((x,a),(y,b)\big)$. Hence $\mathfrak{l}(\hat{f}) = 1$. $\square$

Thus, with the sequence (F_n), we associate the sequence $(\widehat{F}_n)$ of i.i.d. Lipschitz contractions of $\widehat{\mathsf{X}}$ with Lipschitz constants 1. The associated SDS on $\widehat{\mathsf{X}}$ is $(\widehat{X}_n^{x,a})$, as defined in (6.10). From Lemma 2.3, resp., its variant for SDS of contractions, we get the following, where $o \in \mathsf{X}$ and $\hat{o} = (o,1)$.

(6.13) Corollary. $\mathsf{Pr}\big[\hat{d}\big(\widehat{X}_n^{x,a},\hat{o}\big) \to \infty\big] \in \{0,1\}$, *and the value is the same for all* $(x,a) \in \widehat{X}$.

Transient extended SDS

We first consider the situation when $(\widehat{X}_n^{x,a})$ is transient, i.e., the probability in Corollary 6.13 is 1. We shall use the comparison (6.3) of (X_n^x) with the affine stochastic recursion $(Y_n^{|x|})$. Recall that $|x| = d(o,x)$. The hyperbolic extension $(\widehat{Y}_n^{|x|,a})$ of $(Y_n^{|x|})$ is a random walk on the hyperbolic upper half plane. It can be also seen as a random walk on the affine group of all mappings $g_{a,b}(z) = az + b$. Under the non-degeneracy assumptions of Proposition 3.3, this random walk is well-known to be transient.

(6.14) Lemma. *Assume that* (6.1) *and* (6.7) *hold.*
Then for every sufficiently large $r > 0$ *there are* $s > 1$ *and* $\alpha, \delta > 0$ *such that, setting* $K_{r,s} = [0\,,\,r] \times [1/s\,,\,s]$ *and* $Q_{r,\alpha} = [0\,,\,r] \times [\alpha\,,\,\infty)$, *one has for the affine recursion that*

$$\mathsf{Pr}[\widehat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \geq 1] \geq 2\delta \quad \text{for all } (y,a) \in Q_{r,\alpha}\,.$$

Proof.
...
...
...
□

Let $\overline{\mathsf{B}}(r)$ be the closed ball in X with center 0 and radius r. Set $B_{r,s} = \overline{\mathsf{B}}(r) \times [1/s\,,\,s]$ and $C_{r,\alpha} = \overline{\mathsf{B}}(r) \times [\alpha\,,\,\infty)$.

(6.15) Lemma. *Assume that* (6.1) *and* (6.7) *hold and that* $(\widehat{X}_n^{x,a})$ *is transient. Then for every sufficiently large* $r > 0$, *with* α *as in Lemma 6.14,*

$$\mathsf{Pr}[\widehat{X}_n^{x,a} \in C_{r,\alpha} \text{ for infinitely many } n] = 0 \quad \text{for all } (x,a) \in \widehat{X}.$$

Proof. Let

$$\Lambda = \Lambda^{x,a} = \{\omega \in \Omega : \widehat{X}_n^{x,a}(\omega) \in C_{r,\alpha} \text{ for infinitely many } n\}.$$

Given r, let s, α and $\delta > 0$ be as in Lemma 6.14. For each $(c,a) \in Q_{r,\alpha}$ there is an index $N^{c,a} \in \mathbb{N}$ such that

$$\mathsf{Pr}[\widehat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \text{ with } 1 \leq n \leq N^{c,a}] \geq \delta. \tag{6.16}$$

If $(c,a) \notin Q_{r,\alpha}$ then we set $N^{c,a} = 0$. Since $B_{r,s}$ is compact, the transience assumption yields that $\mathsf{Pr}\big(\bigcup_{j=2}^{\infty} \Omega_j\big) = 1$, where

$$\Omega_j = \Omega_j^{x,a} = \{\omega \in \Omega : \widehat{X}_n^{x,a}(\omega) \notin B_{r,s} \text{ for every } n \geq j\}.$$

Thus, we need to show that $\mathsf{Pr}(\Lambda \cap \Omega_j) = 0$ for every $j \geq 2$. We define a sequence of stopping times $\tau_k = \tau_k^{x,a}$ and (when $\tau_k < \infty$) associated pairs $(x_k, a_k) = \widehat{X}_{\tau_k}^{x,a}$ by

$$\tau_1 = \inf\{n > N^{|x|,a} : \widehat{X}_n^{x,a} \in C_{r,\alpha}\} \quad \text{and}$$

$$\tau_{k+1} = \begin{cases} \inf\{n > \tau_k + N^{|x_k|,a_k} : \widehat{X}_n^{x,a} \in C_{r,\alpha}\}\,, & \text{if } \tau_k < \infty\,,\\ \infty\,, & \text{if } \tau_k = \infty\,.\end{cases}$$

Unless explained separately, we always use $\tau_k = \tau_k^{x,a}$. Note that $\omega \in \Lambda$ if and only if $\tau_k(\omega) < \infty$ for all k. Therefore

$$\Lambda \cap \Omega_j = \bigcap_{k\geq j} \Lambda_{j,k}\,, \quad \text{where} \quad \Lambda_{j,k} = [\tau_k < \infty\,,\ \widehat{X}_n^{x,a} \notin B_{r,s} \text{ for all } n \text{ with } j \leq n \leq \tau_k].$$

We have $\Lambda_{j,k} \subset \Lambda_{j,k-1}$. Next, note that

$$\text{if} \quad \widehat{X}_n^{x,a}(\omega) \notin B_{r,s} \quad \text{then} \quad \widehat{Y}_n^{|x|,a}(\omega) \notin K_{r,s}\,.$$

This follows from (6.3).

We have that $\widehat{X}_{\tau_{k-1}}^{x,a} \in C_{r,\alpha}$ for $k \geq 2$. Just for the purpose of the next lines of the proof, we introduce the measure σ on $C_{r,\alpha}$ given by $\sigma(\widehat{B}) = \mathsf{Pr}\big(\Lambda_{j,k-1} \cap [\widehat{X}_{\tau_{k-1}}^{x,a} \in \widehat{B}]\big)$, where $\widehat{B} \subset C_{r,\alpha}$ is a Borel set. Then, using the strong Markov property and (6.16),

$$\begin{aligned}
\mathsf{Pr}(\Lambda_{j,k}) &= \mathsf{Pr}\Big([\tau_k < \infty\,,\ \widehat{X}_n^{x,a} \notin B_{r,s} \text{ for all } n \text{ with } \tau_{k-1} < n \leq \tau_k] \cap \Lambda_{j,k-1}]\Big)\\
&= \int_{C_{r,\alpha}} \mathsf{Pr}[\tau_1^{y,b} < \infty\,,\ \widehat{X}_n^{y,b} \notin B_{r,s} \text{ for all } n \text{ with } 0 < n \leq \tau_1^{y,b}]\, d\sigma(y,b)\\
&\leq \int_{C_{r,\alpha}} \mathsf{Pr}[\tau_1^{y,b} < \infty\,,\ \widehat{Y}_n^{|y|,b} \notin K_{r,s} \text{ for all } n \text{ with } 0 < n \leq N^{|y|,b}]\ d\sigma(y,b)\\
&\leq \int_{C_{r,\alpha}} (1-\delta)\ d\sigma(y,b) = (1-\delta)\,\mathsf{Pr}(\Lambda_{j,k-1})\,.
\end{aligned}$$

We continue recursively downwards until we reach $k = 2$ (since $k = 1$ is excluded unless $(x,a) \in C_{r,\alpha}$). Thus, $\mathsf{Pr}(\Lambda_{j,k}) \leq (1-\delta)^{k-1}$, and as $k \to \infty$, we get $\mathsf{Pr}(\Lambda \cap \Omega_j) = 0$, as required. □

(6.17) Theorem. *Given the random i.i.d. Lipschitz mappings F_n, let A_n and $|B|_n$ be as in* (6.2). *Suppose that* (6.1) *and* (6.7) *hold, and that* $\mathsf{Pr}\big[\hat{d}\big(\widehat{X}_n^{x,a}, \hat{o}\big) \to \infty\big] = 1$. *Then the SDS induced by the F_n on X is locally contractive.*

In particular, it has an invariant Radon measure ν that is unique up to multiplication with constants.

Also, the shift T on $\big(\mathsf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathsf{X}^{\mathbb{N}_0}), \mathsf{Pr}_\nu\big)$ is ergodic, where Pr_ν is the measure on $\widehat{\mathsf{X}}^{\mathbb{N}_0}$. associated with ν.

Proof. Fix any starting point (x,a) of the extended SDS. Let r be sufficiently large so that the last two lemmas apply, and such that

$$\mathsf{Pr}[X_n^x \in \overline{\mathsf{B}}(r) \text{ for infinitely many } n] = 1.$$

We claim that

$$\lim_{n\to\infty} A_{0,n}\, \mathbf{1}_{\overline{\mathsf{B}}(r)}(X_n^x) = 0 \quad \text{almost surely.} \tag{6.18}$$

We know by transience of the extended SDS that for every $s \ge 1$

$$\mathsf{Pr}[\widehat{X}_n^{x,a} \in B_{r,s} \text{ for infinitely many } n] = 0.$$

We combine this with Lemma 6.14 and consider α associated with r as in that lemma: by Lemma 6.15,

$$\mathsf{Pr}[\widehat{X}_n^{x,a} \in B_{r,s} \cup C_{r,\alpha} \text{ for infinitely many } n] = 0.$$

If $s \ge \alpha$, then $B_{r,s} \cup C_{r,\alpha} = \overline{\mathsf{B}}(r) \times [1/s\,,\,\infty)$.

Thus, if $\mathbb{N}(x,r)$ denotes the a.s. infinite random set of all n for which $X_n^x \in \overline{\mathsf{B}}(r)$, then for all but finitely many $n \in \mathbb{N}(x,r)$, we have $A_{0,n} < 1/s$. We have proved (6.18). We conlcude that

$$d(X_n^x\,,X_n^y)\,\mathbf{1}_{\overline{\mathsf{B}}(r)}(X_n^x) \le A_{0,n}\, d(x,y)\,\mathbf{1}_{\overline{\mathsf{B}}(r)}(X_n^x) \to 0 \quad \text{almost surely.}$$

Now that we have local contractivity, the remaining statements follow from Theorem 2.13. □

Non-transient extended SDS

Now we assume to be in the non-transient case, i.e., the probability in Corollary 6.13 is 0. We start with an invariant measure ν for the SDS on X. If (6.1) & (6.7) hold, its existence is guaranteed by Corollary 6.8. Then we extend ν to a measure $\lambda = \lambda_\nu$ on $\widehat{\mathsf{X}}$, as follows.

In the non-lattice case,

$$\int_{\widehat{\mathsf{X}}} \varphi(x,a)\, d\lambda(x,a) = \int_{\mathsf{X}} \int_{\mathbb{R}} \varphi(x,e^u)\, d\nu(x)\, du\,.$$

This is the product of ν with the multiplicative Haar measure on $\mathbb{R}^+$.

In the lattice case,

$$\int_{\widehat{\mathsf{X}}} \varphi(x,a)\, d\lambda(x,a) = \int_{\mathsf{X}} \sum_{m\in\mathbb{Z}} \varphi(x,e^{\kappa m})\, d\nu(x)\,.$$

In both cases, it is straightforward to check that λ is an invariant Radon measure for the extended SDS on the respective extended space $\widehat{\mathsf{X}}$. We can realize the latter SDS, starting at $(x,a) \in \widehat{\mathsf{X}}$, on the space

$$\bigl(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathfrak{B}(\widehat{\mathsf{X}}^{\mathbb{N}_0}), \mathsf{Pr}_{x,a}\bigr),$$

where $\mathfrak{B}(\widehat{\mathsf{X}}^{\mathbb{N}_0})$ is the product Borel σ-algebra, and $\mathsf{Pr}_{x,a}$ is the image of the measure Pr under the mapping

$$\Omega \to \widehat{\mathsf{X}}^{\mathbb{N}_0}\,, \quad \omega \mapsto \bigl(\widehat{X}_n^{x,a}(\omega)\bigr)_{n\ge 0}\,.$$

Then we consider the Radon measure on $\widehat{\mathsf{X}}^{\mathbb{N}_0}$ defined by

$$\mathsf{Pr}_\lambda = \int_{\widehat{\mathsf{X}}} \mathsf{Pr}_{x,a}\, d\lambda(x,a).$$

The integral with respect to Pr_λ is denoted E_λ. We write $\widehat{T}$ for the time shift on $\widehat{\mathsf{X}}^{\mathbb{N}_0}$. Since λ is invariant for the extended SDS, $\widehat{T}$ is a contraction of $L^1(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathsf{Pr}_\lambda)$. Also, in this section, $\mathfrak{J}$ stands for the σ-algebra of the $\widehat{T}$-invariant sets in $\mathfrak{B}(\widehat{\mathsf{X}}^{\mathbb{N}_0})$. As before, any function $\varphi : \widehat{\mathsf{X}}^{\ell} \to \mathbb{R}$ is extended to $\widehat{\mathsf{X}}^{\mathbb{N}_0}$ by setting $\varphi(\mathbf{x}, \mathbf{a}) = \varphi\big((x_0, a_0), \dots, (x_{\ell-1}, a_{\ell-1})\big)$, if $(\mathbf{x}, \mathbf{a}) = \big((x_n, a_n)\big)_{n \geq 0}$.

In analogy with (2.4), we define

$$\widehat{X}^{x,a}_{m,n} = \big(X^x_{m,n}\,,\, A_{m,n}a\big) \quad (n \geq m)\,.$$

We now set for $n \geq m$ and $\varphi : \widehat{\mathsf{X}}^{\mathbb{N}_0} \to \mathbb{R}$

$$S^{x,a}_{m,n}\varphi(\omega) = \sum_{k=m}^{n} \varphi\Big(\big(\widehat{X}^{x,a}_{m,k}(\omega)\big)_{k \geq m}\Big)$$

and in particular $S^{x,a}_n\varphi(\omega) = S^{x,a}_{0,n}\varphi(\omega)$. Consider the sets

$$\Omega_r = \big\{\omega \in \Omega : \liminf \hat{d}\big(\widehat{X}^{\hat{o}}_n(\omega), \hat{o}\big) \leq r\big\} \ \ (r \in \mathbb{N}) \quad \text{and} \quad \Omega_\infty = \bigcup_r \Omega_r\,. \tag{6.19}$$

By our assumption of non-transience, $\mathsf{Pr}(\Omega_\infty) = 1$. For $r \in \mathbb{N}$, write $\widehat{B}(r)$ for the *closed* ball in $(\widehat{\mathsf{X}}, \hat{d})$ with center $\hat{o}$ and radius r. Then for every $\omega \in \Omega_r$ and $s \in \mathbb{N}_0$, the set $\{n : \widehat{X}^{x,a}_n(\omega) \in \widehat{B}(r+s)$ for all $(x,a) \in \widehat{B}(s)\}$ is infinite. For each r, set $\psi_r(x,a) = \max\big\{1 - \hat{d}\big((x,a), \widehat{B}(r)\big)\,,\, 0\big\}$. Then $\psi_r \in \mathcal{C}^+_c(\widehat{\mathsf{X}})$ satisfies

$$\begin{aligned} &\mathbf{1}_{\widehat{B}(r+1)} \geq \psi_r \geq \mathbf{1}_{\widehat{B}(r)}\,,\\ &|\psi(x,a) - \psi(y,b)| \leq \hat{d}\big((x,a),(y,b)\big) \text{ on } \widehat{\mathsf{X}}\,, \quad \text{and}\\ &S^{x,a}_n\psi_{r+s}(\omega) \to \infty \quad \text{for all } \omega \in \Omega_r\,,\ (x,a) \in \widehat{B}(s)\,. \end{aligned} \tag{6.20}$$

Then we can find a decreasing sequence of numbers $c_r > 0$ such that $\sum_r c_r \max \psi_{r+2} < \infty$ and the functions

$$\Phi = \sum_r c_r\, \psi_{r+2} \quad \text{and} \quad \Psi = \sum_r c_r\, \psi_r \tag{6.21}$$

are in $L^1(\widehat{\mathsf{X}}, \lambda)$ and thus (there extensions to $\mathsf{X}^{\mathbb{N}_0}$) in $L^1(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathsf{Pr}_\lambda)$. They will be used below several times. Both are continuous and strictly positive on $\widehat{\mathsf{X}}$, and by construction,

$$\sum_n \Psi\big(\widehat{X}^{x,a}_n(\omega)\big) = \infty \quad \text{for all } \omega \in \Omega_\infty \text{ and } (x,a) \in \widehat{\mathsf{X}}\,.$$

We have obtained the following.

(6.22) Lemma. *When the extended SDS is non-transient, T is conservative.*

Next, for any $\varphi \in L^1(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathsf{Pr}_\lambda)$, consider the function $\mathbf{v}_\varphi = \mathsf{E}_\lambda(\varphi \mid \mathfrak{J})/\mathsf{E}_\lambda(\Psi \mid \mathfrak{J})$ on $\widehat{\mathsf{X}}^{\mathbb{N}_0}$. A priori, the quotient of conditional expectations is defined only Pr_λ-almost everywhere, and we consider a representative which is always finite. We turn this into the family of finite positive random variables

$$V^{x,a}_\varphi(\omega) = \mathbf{v}_\varphi\Big(\big(\widehat{X}^{x,a}_n(\omega)\big)_{n \geq 0}\Big), \quad (x,a) \in \widehat{\mathsf{X}}.$$

(6.23) Lemma. *In the non-transient case, let $\tau : \Omega \to \mathbb{N}$ be any a.s. finite random time. Then, on the set where $\tau(\omega) < \infty$, for every $\varphi \in L^1(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathsf{Pr}_\lambda)$,*

$$\lim_{n\to\infty} \frac{S_n^{x,a}\varphi - S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi - S_\tau^{x,a}\Psi} = V_\varphi^{x,a} \quad \mathsf{Pr}\text{-}almost\ surely,\ for\ \lambda\text{-}almost\ every\ (x,a) \in \widehat{\mathsf{X}}.$$

Proof. We know that $S_n^{x,a}\Psi(\omega) \to \infty$ for all $\omega \in \Omega_\infty$. Once more by the Chacon-Ornstein theorem, $S_n^{x,a}\varphi/S_n^{x,a}\Psi \to V_\varphi^{x,a}$ almost surely on Ω_∞, for λ-almost every $(x,a) \in \widehat{\mathsf{X}}$. Furthermore, both $S_\tau^{x,a}\varphi/S_n^{x,a}\Psi$ and $S_\tau^{x,a}\Psi/S_n^{x,a}\Psi$ tend to 0 on Ω_∞, as $n \to \infty$. When $n > \tau$,

$$\frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} = \underbrace{\frac{S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi}}_{\to\, 0 \text{ a.s.}} + \Biggl(1 - \underbrace{\frac{S_\tau^{x,a}\Psi}{S_n^{x,a}\Psi}}_{\to\, 0 \text{ a.s.}}\Biggr) \frac{S_n^{x,a}\varphi - S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi - S_\tau^{x,a}\Psi}.$$

The statement follows. □

When the extended SDS is non-transient, we do not see how to involve local contractivity, but we can provide a reasonable additional assumption which will yield uniqueness of the invariant Radon measure. We set

$$D_n(x,y) = \frac{d(X_n^x, X_n^y)}{A_1 \cdots A_n}. \tag{6.24}$$

(Compare with the proof of Theorem 4.2, which corresponds to $A_n \equiv 1$.) The assumption is

$$\mathsf{Pr}[D_n(x,y) \to 0] = 1 \quad \text{for all } x, y \in \mathsf{X}. \tag{6.25}$$

(6.26) Remark. If we set $D_{m,n}(x,y) = d(X_{m,n}^x, X_{m,n}^y)/A_{m,n}$ then (6.25) implies that

$$\mathsf{Pr}\Bigl[\lim_{n\to\infty} D_{m,n}(x,y) = 0 \text{ for all } x, y \in \mathsf{X},\ m \in \mathbb{N}\Bigr] = 1.$$

Indeed, let X_0 be a countable, dense subset of X. Then (6.25) implies that

$$\mathsf{Pr}\Bigl[\lim_{n\to\infty} D_{m,n}(x,y) = 0 \text{ for all } x, y \in \mathsf{X}_0,\ m \in \mathbb{N}\Bigr] = 1.$$

Let Ω_0 be the subset of Ω_∞ where this holds.

Note that $D_{m,n}(x,y) \le d(x,y)$. Given arbitrary $x, y \in \mathsf{X}$ and $x_0, y_0 \in \mathsf{X}_0$, we get on Ω_0

$$D_{m,n}(x,y) \le D_{m,n}(x_0,y_0) + d(x,x_0) + d(y,y_0),$$

and the statement follows. □

In the next lemma, we give a condition for (6.25). It will be useful, in §7.

(6.27) Lemma. *In the case when the extended SDS is non-transient, suppose that for every $\varepsilon > 0$ and $r \in \mathbb{N}$ there is k such that $\mathsf{Pr}[D_k(x,y) < \varepsilon$ for all $x, y \in \mathsf{B}(r)] > 0$. for all $x, y \in \mathsf{X}$. Then (6.25) holds.*

Proof. We set $D_\infty(x,y) = \lim_n D_n(x,y)$ and $w(x,y) = \mathsf{E}\bigl(D_\infty(x,y)\bigr)$. A straightforward adaptation of the argument used in the proof of Theorem 4.2 yields that

$$\lim_{m\to\infty} \frac{w(X_m^x, X_m^y)}{A_1 \cdots A_m} = D_\infty(x,y) \quad \text{almost surely.} \tag{6.28}$$

Again, we claim that $\Pr[D_\infty(x,y) \geq \varepsilon] = 0$. By non-transience, it is sufficient to show that $\Pr(W_r) = 0$ for every $r \in \mathbb{N}$, where

$$W_r = \bigcap_{m \geq k} \bigcup_{n \geq m} [\widehat{X}_n^x\,,\ \widehat{X}_n^y \in \mathsf{B}(r) \times [1/r\,,\,r]\,,\ D_n(x,y) \geq \varepsilon]\,.$$

By assumption, there is k such that the event $C_{k,r} = [D_k(x,y) < \varepsilon/2 \text{ for all } x,y \in \mathsf{B}(r)]$ satisfies $\Pr(C_{k,r}) > 0$.

We now continue as in the proof of Theorem 4.2, and find that for all $u,v \in \mathsf{B}(r)$ with $d(u,v) \geq \varepsilon$,

$$w(u,v) \leq d(u,v) - \delta\,, \quad \text{where} \quad \delta = \Pr(C_{k,r}) \cdot (\varepsilon/2) > 0.$$

This yields that on W_r, almost surely we have infinitely many $n \geq k$ for which $w(X_n^x, X_n^y) \leq d(X_n^x, X_n^y) - \delta$ and $A_1 \cdots A_n \leq r$, that is,

$$\frac{w(X_n^x, X_n^y)}{A_1 \cdots A_n} \leq D_n(x,y) - \frac{\delta}{r} \quad \text{infinitely often.}$$

Letting $n \to \infty$, we get $D_\infty(x,y) < D_\infty(x,y)$ almost surely on W_r, so that indeed $\mathsf{Pr}(W_r) = 0$. $\square$

We now elaborate the main technical prerequisite for handling the case when the extended SDS in non-transient. Some care may be in place to have a clear picture regarding the dependencies of sets on which various "almost everywhere" statements hold. Let $\varphi \in L^1(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathsf{Pr}_\lambda)$. For λ-almost every $(x,a) \in \widehat{\mathsf{X}}$, there is a set $\Omega_\varphi^{x,a} \subset \Omega_0$ with $\mathsf{Pr}(\Omega_\varphi^{x,a}) = 1$, such that

$$\frac{S_n^{x,a}\varphi(\omega)}{S_n^{x,a}\Psi(\omega)} \to V_\varphi^{x,a}(\omega)$$

for every $\omega \in \Omega_\varphi^{x,a}$. For the remaining $(x,a) \in \widehat{\mathsf{X}}$, we set $\Omega_\varphi^{x,a} = \emptyset$.

(6.29) Proposition. *In the case when the extended SDS is non-transient, assume* (6.25). *Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathsf{X}}^\ell)$ with $\ell \geq 1$. Then for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \varphi) > 0$ with the following property.*

For all $(x,a),(y,b) \in \widehat{\mathsf{X}}$ and any a.s. finite random time $\tau : \Omega \to \mathbb{N}_0$, one has on the set of all $\omega \in \Omega_\Phi^{x,a}$ with $\tau(\omega) < \infty$ and $\bigl|\log\bigl(A_{0,\tau}(\omega)a/b\bigr)\bigr| < \delta$ that

$$\limsup_{n\to\infty} \left| \frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} - \frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi} \right| \leq \varepsilon\, W^{x,a}\,,$$

where $W^{x,a} = V_\Phi^{x,a} + 1$.

Proof. Recall that Φ, Ψ, φ and ψ_r are also considered as functions on $\mathsf{X}^{\mathbb{N}_0}$ via their extensions defined above.

Since Ψ is continuous and > 0, there is $C = C_\varphi > 0$ such that $\varphi \leq C \cdot \Psi$. Also, there is some $r_0 \in \mathbb{N}$ such that the projection of $\mathsf{supp}(\varphi)$ onto the first coordinate (i.e., the one with index 0) is contained in $B(r_0)$. We let $\varepsilon' = \min\{\varepsilon/2, \varepsilon/(2C), c_{r_0+1}\varepsilon/2, 1\}$, where c_{r_0+1} comes from the definition (6.21) of Φ and Ψ. Since φ is uniformly continuous, there is $\delta > 0$ with $2\delta \le \varepsilon'$ such that

$$\big|\varphi\big((x_0,a_0),\dots,(x_{\ell-1},a_{\ell-1})\big) - \varphi\big((y_0,b_0),\dots,(y_{\ell-1},b_{\ell-1})\big)\big| \le \varepsilon'$$
$$\text{whenever}\quad \hat d\big((x_j,a_j),(y_j,b_j)\big) < 2\delta\,,\; j=0,\dots,\ell-1.$$

We write

$$\left|\frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} - \frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi}\right| \le \underbrace{\frac{|S_n^{x,a}\varphi - S_{\tau,n}^{y,b}\varphi|}{S_n^{x,a}\Psi}}_{\text{Term 1}} + \underbrace{\frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi}}_{\le C_\varphi}\underbrace{\frac{|S_n^{x,a}\Psi - S_{\tau,n}^{y,b}\Psi|}{S_n^{x,a}\Psi}}_{\text{Term 2}}.$$

We consider the random element $z = X_\tau^x$, so that $X_n^x = X_{\tau,n}^z$. Using the dilation invariance of hyperbolic metric,

$$\begin{aligned}\hat d(\widehat X_n^{x,a}\,,\widehat X_{\tau,n}^{y,b}) &= \theta\big(\mathfrak{i}\,A_{0,n}a\,,\, d(X_{\tau,n}^z\,,X_{\tau,n}^y) + \mathfrak{i}\,A_{\tau,n}b\big)\\ &= \theta\big(\mathfrak{i}\,A_{0,\tau}a\,,D_{\tau,n}(z,y)+\mathfrak{i}\,b\big) \le |\log(A_{0,\tau}a/b)| + D_{\tau,n}(z,y)+\mathfrak{i}\,b\big)\,.\end{aligned}$$

By (6.25), for $\omega \in \Omega_\Phi^{x,a}$ with $\tau(\omega) < \infty$ there is a finite $\sigma(\omega) \ge \tau(\omega)$ in $\mathbb{N}$ such that $\theta\big(\mathfrak{i}\,a, D_{\tau,n}(z,y)+\mathfrak{i}\,a\big) < \delta$ for all $n \ge \sigma(\omega)$. In the sequel, we assume that our $\omega \in \Omega_\Phi^{x,a}$ also satisfies $\big|\log\big(A_{0,\tau}(\omega)a/b\big)\big| < \delta$.

Now, we first bound the lim sup of Term 1 by $\varepsilon/2$. If $n \ge \sigma$ and $|A_{0,\tau}(\omega)a/b| < \delta$, then we obtain that

$$\big|\varphi\big(\widehat X_n^{x,a}\,,\widehat X_{n+1}^{x,a}\,,\dots,\widehat X_{n+\ell-1}^{x,a}\big) - \varphi\big(\widehat X_{\tau,n}^{y,b}\,,\widehat X_{\tau,n+1}^{y,b}\,,\dots,\widehat X_{\tau,n+\ell-1}^{y,b}\big)\big| < \varepsilon' \le \varepsilon/2.$$

Suppose in addition that at least one of the two values $\varphi\big(\widehat X_n^{x,a},\widehat X_{n+1}^{x,a}\,,\dots,\widehat X_{n+\ell-1}^{x,a}\big)$ or $\varphi\big(\widehat X_{\tau,n}^{y,b}\,,\widehat X_{\tau,n+1}^{y,b}\,,\dots,\widehat X_{\tau,n+\ell-1}^{y,b}\big)$ is positive. Then at least one of $\widehat X_n^{x,a}$ or $\widehat X_{\tau,n}^{y,b}$ belongs to $\widehat B(r_0)$, and by the above (since $\delta < 1$) both belong to $\widehat B(r_0+1)$. Thus, for $n \ge \sigma$,

$$\begin{aligned}\big|\varphi\big(\widehat X_n^{x,a}\,,\widehat X_{n+1}^{x,a}\,,\dots,\widehat X_{n+\ell-1}^{x,a}\big) - \varphi\big(\widehat X_{\tau,n}^{y,b}\,,\widehat X_{\tau,n+1}^{y,b}\,,\dots,\widehat X_{\tau,n+\ell-1}^{y,b}\big)\big| &\le \varepsilon'\,\psi_{r_0+1}\big(\widehat X_n^{x,a}\big)\\ &\le (\varepsilon/2)\,\Psi\big(\widehat X_n^{x,a}\big).\end{aligned}$$

We get

$$\frac{\big|\big(S_n^{x,a}\varphi - S_\sigma^{x,a}\varphi\big) - \big(S_{\tau,n}^{y,b}\varphi - S_{\tau,\sigma}^{y,b}\varphi\big)\big|}{S_n^{x,a}\Psi - S_\sigma^{x,a}\Psi} \le \varepsilon/2.$$

Since $S_n^{x,a}\Psi \to \infty$ almost surely, when passing to the lim sup, we can omit all terms in the last inequality that contain a σ; see Lemma 6.23. This yields the bound on the lim sup of Term 1.

Next, we bound the lim sup of Term 2 by $\varepsilon/2$. We start in the same way above, replacing φ with an arbitrary one among the functions ψ_r and replacing ℓ with 1. Using the specific properties (6.20) of ψ_r (in particular, Lipschitz continuity with constant 1), and replacing $\widehat B(r_0)$ with $\widehat B(r+1) = \mathsf{supp}(\psi_r)$, we arrive at the inequality

$$\big|\psi_r\big(\widehat X_n^{x,a}\big) - \psi_r\big(\widehat X_{\tau,n}^{y,b}\big)\big| \le \frac{\varepsilon}{2C}\,\psi_{r+2}\big(\widehat X_n^{x,a}\big).$$

It holds for all $n \ge \sigma$, with probability 1. We deduce

$$\big|\Psi\big(\widehat X_n^{x,a}\big) - \Psi\big(\widehat X_{\tau,n}^{y,b}\big)\big| \le \frac{\varepsilon}{2C}\,\Phi\big(\widehat X_n^{x,a}\big)$$

and

$$\frac{\left|\left(S_n^{x,a}\Psi - S_\sigma^{x,a}\Psi\right) - \left(S_{\tau,n}^{y,b}\Psi - S_{\tau,\sigma}^{y,b}\Psi\right)\right|}{S_n^{x,a}\Psi - S_\sigma^{x,a}\Psi} \le \frac{\varepsilon}{2C}\,\frac{S_n^{x,a}\Phi - S_\sigma^{x,a}\Phi}{S_n^{x,a}\Psi - S_\sigma^{x,a}\Psi}$$

Passing to the lim sup as above, and using the Chacon-Ornstein theorem here, we get that the lim sup of Term 2 is bounded almost surely by $\frac{\varepsilon}{2C}\, V_\Phi^{x,a}$. $\square$

In the sequel, when we sloppily say "for almost every $a > 0$", we shall mean "for Lebesgue-almost every $a > 0$" in the non-lattice case, resp. "for every $a = e^{-\kappa m}$ $(m \in \mathbb{Z})$" in the lattice case.

(6.30) Corollary. *Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathsf{X}}^\ell)$ as above. For almost every $a > 0$, there is a set $\Omega_\varphi^a \subset \Omega_0$ with $\mathsf{Pr}(\Omega_\varphi^a) = 1$ such that for all $x, y \in \mathsf{X}$,*

$$V_\varphi^{x,a} = V_\varphi^{y,a} =: V_\varphi^a.$$

Proof. For almost every a, there is at least one $x_a \in \mathsf{X}$ such that $\mathsf{Pr}(\Omega_\varphi^{x_a,a}) = 1$. We can apply Proposition 6.29 with arbitrary $y \in \mathsf{X}$, $b = a$ and $\tau = 0$. Then we are allowed to take any $\varepsilon > 0$ and get that $V_\varphi^{x,a} = V_\varphi^{y,a}$ on $\Omega_\varphi^{x_a,a} \cap \Omega_\Phi^{x_a,a}$. $\square$

(6.31) Proposition. *Suppose that* (6.1), (6.7) *and* (6.25) *hold, and that the extended SDS is non-transient. Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathsf{X}}^\ell)$, as above. Then for almost every $a > 0$, the random variable V_φ^a is almost surely constant (depending on φ and – so far – on a).*

Proof. Let a be such that $\Pr(\Omega_\varphi^a) = 1$, and choose $x = x_a$ as in the proof of Corollary 6.30.

For $s \in \mathbb{N}$, let $\varepsilon_s = 1/s$ and $\delta_s = \delta(\varepsilon_s\,, \varphi)$ according to Proposition 6.29. By our assumptions, $(A_{0,n})_{n\ge1}$ is a topologically recurrent random walk on $\mathbb{R}^+$, starting at 1. Choose $m \in \mathbb{N}$ and let $\tau_{m,s}$ be the m-th return time to the interval $(e^{-\delta_s}\,, e^{\delta_s})$. For every m and s, this is an almost surely finite stopping time, and we can find $\bar\Omega_\varphi^a \subset \Omega_\varphi^a \cap \Omega_\Phi^{x,a}$ with $\mathsf{Pr}(\bar\Omega_\varphi^a) = 1$ such that all $\tau_{m,s}$ are finite on that set.

We now apply Proposition 6.29 with $(y, b) = (x, a)$ and $\tau = \tau_{m,s}$. Then

$$\limsup_{n\to\infty}\Bigg|V^a\varphi - \underbrace{\frac{S_{\tau,n}^{x,a}\varphi}{S_{\tau,n}^{x,a}\Psi}}_{=:\,U_{n,m,s}}\Bigg| \le \frac{1}{s}\,W^{x,a}\,,$$

Since our stopping time satisfies $\tau_{m,s} \ge m$, the random variable $U_{n,m,s}$ (depending also on φ and (x,a)) is independent of the basic random mappings $F_1, \dots, F_m$. (Recall that the F_k that appear in $S_{\tau,n}^{x,a}$ are such that $k \ge \tau + 1$.) We get

$$\lim_{s\to\infty}\limsup_{n\to\infty} |V^a\varphi - U_{n,m,s}| = 0$$

on $\bar\Omega_\varphi^a$. Therefore also $V^a\varphi$ is independent of $F_1, \dots, F_m$. This holds for every m. By Kolmogorov's 0-1-law, $V^a\varphi$ is almost surely constant.

Note that in the lattice case, the proof simplifies, because we can just take τ to be the first return time of $A_{0,n}$ to 1. $\square$

(6.32) Theorem. *Given the random i.i.d. Lipschitz mappings F_n, let A_n and $|B|_n$ be as in* (6.2). *Suppose that* (6.1), (6.7) *and* (6.25) *hold, and that* $\mathsf{Pr}\big[\hat d\big(\widehat X_n^{x,a}, \hat o\big) \to \infty\big] = 0$.

Then the SDS induced by the F_n *on* X *has an invariant Radon measure* ν *that is unique up to multiplication with constants.*

Also, the shift $\widehat{T}$ *on* $\big(\widehat{\mathsf{X}}^{\mathbb{N}_0}, \mathfrak{B}(\widehat{\mathsf{X}}^{\mathbb{N}_0}), \mathsf{Pr}_\lambda\big)$ *is ergodic, where* λ *is the extension of* ν *to* $\widehat{\mathsf{X}}$ *and* Pr_λ *the associated measure on* $\widehat{\mathsf{X}}^{\mathbb{N}_0}$.

Proof. Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathsf{X}}^\ell)$. Recall that the function $\mathbf{v}_\varphi = \mathsf{E}_\lambda(\varphi \mid \mathfrak{I})/\mathsf{E}_\lambda(\Psi \mid \mathfrak{I})$ on $\widehat{\mathsf{X}}^{\mathbb{N}_0}$ is T-invariant. For the random variables $V_\varphi^{x,a} = V_\varphi^a$, this means that for almost every $a > 0$,

$$V_\varphi^a = V_\varphi^{A_{0,n}a} \quad \text{Pr-almost surely for all } n\,.$$

By Proposition 6.31, these random variables are constant on a set $\bar\Omega_\varphi^a \subset \Omega_\varphi^a$ with $\Pr(\bar\Omega_\varphi^a) = 1$. Fix one $a_0 > 0$ for which this holds.

In the lattice case, since we have chosen the maximal κ for which $\log A_n \in \kappa \cdot \mathbb{Z}$ a.s., the associated centered random walk $\log A_{0,n}$ is recurrent on $\kappa \cdot \mathbb{Z}$: for every starting point $a \in \exp(\kappa \cdot \mathbb{Z})$, we have that $(A_{0,n}a)_{n\ge 0}$ visits a_0 almost surely. We infer that $V_\varphi^a = V_\varphi^{a_0}$ Pr-almost surely for every $a \in \exp(\kappa \cdot \mathbb{Z})$.

In the non-lattice case, the multiplicative random walk $(A_{0,n}a)_{n\ge 0}$ starting at any $a > 0$ is topologically recurrent on $\mathbb{R}^+$. This means that for every $a > 0$, with probability 1 there is a random sequence $(n_k)_{k\ge 0}$ such that $A_{0,n_k}a \to a_0$ as $k \to \infty$. Proposition 6.29 yields that $V_\varphi^a = V_\varphi^{a_0}$ on a set $\widetilde\Omega_\varphi^a \subset \Omega_\varphi^{a_0}$ with probability 1.

Now let $\{a_k : k \in \mathbb{N}\}$ be dense in $\mathbb{R}^+$ and such that $\Pr(\widetilde\Omega_\varphi^{a_k}) = 1$ for all $\mathbb{N}$. Using Proposition 6.29 once more, we get that for every $a > 0$, $V_\varphi^a = V_\varphi^{a_k} = V_\varphi^{a_0}$ on $\bigcap_k \widetilde\Omega_\varphi^{a_k}$.

We conclude that $\mathbf{v}_\varphi$ is constant Pr_λ-almost surely.

This is true for any $\varphi \in \mathcal{C}_c^+(\widehat{\mathsf{X}}^\ell)$. Therefore $\widehat{T}$ is ergodic. It follows that up to multiplication with constants, λ is the unique invariant measure on $\widehat{\mathsf{X}}$ for the extended SDS, so that ν is the unique invariant measure on X for the original SDS. By Corollary 6.8(b), $\mathsf{supp}(\nu) = \mathsf{L}$. $\square$

We remark that by projecting, also the shift T on $\big(\mathsf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathsf{X}^{\mathbb{N}_0}), \mathsf{Pr}_\nu\big)$ is ergodic.

7. The reflected affine stochastic recursion

We finally consider in detail the SDS of (1.1). Thus, $F_n(x) = |A_n x - B_n|$, so that $\mathfrak{l}(F_n) = A_n$ and $|B|_n = |B_n|$. We assume (6.1).

In the case when $\mathsf{E}(\log A_n) < 0$, we can once more apply Propositions 2.21, resp. 3.2, and Corollary 6.4.

(7.1) Corollary. *If* $\mathsf{E}(\log^+ A_n) < \infty$ *and* $-\infty \le \mathsf{E}(\log A_n) < 0$ *then the reflected affine stochastic recursion is strongly contractive on* $[0\,,\,\infty)$.

If in addition $\mathsf{E}(\log^+ |B_n|) < \infty$ *then it has a unique invariant probability measure* ν *on* $[0\,,\,\infty)$*, and it is (positive) recurrent on* $\mathsf{L} = \mathsf{supp}(\nu)$.

From now on, we shall be interested in the case when $\log A_n$ *is centered.*

For the time being, we shall only deal with the case when $B_n \ge 0$. We can use Remark 2.10; compare with the arguments used after Corollary 6.4. Thus, the reflected affine stochastic recursion is topologically irreducible on the set L given by Corollary 2.9. Here, we shall not investigate the nature of L in detail. It may be unbounded or compact, and even finite.

Since we have $\mathsf{X} = [0\,,\,\infty)$, the extended space $\widehat{\mathsf{X}}$ is just the first quadrant with hyperbolic metric, and if $f(x) = |ax - b|$ then $\hat{f}(x,y) = (|ax-b|, ay)$. We can apply Corollary 6.13 to the extended process.

(7.2) Proposition. *Assume that* (6.1) *holds, and that* $\mathsf{E}(|\log A_n|) < \infty$, $\mathsf{E}(\log A_n) = 0$, $B_n > 0$ *almost surely, and* $\mathsf{E}(\log^+ B_n) < \infty$.

If the extended process $(\widehat{X}_n^{x,a})$ *is non-transient, then the normalized distances* $D_n(x,y)$ *of* (6.24) *satisfy* (6.25), *that is,* $\mathsf{Pr}[d(Z_n^x, Z_n^y) \to 0] = 1$ *for all* $x, y \in \mathsf{X}$, *where* $Z_n^x = X_n/A_{0,n}$.

Proof. We have the recursion $Z_0^x = x$ and $Z_n^x = |Z_{n-1}^x - B_n/A_{0,n}|$. We start with a simple exercise whose proof we omit. Let $c_j > 0$ and $f_j(x) = |x - c_j|$, $j = 1, \dots, s$. Then

$$f_s \circ \dots \circ f_2 \circ f_1(x) \le \max\{c_1\,, \dots, c_s\} \quad \text{for all} \quad x \in [0\,,\, c_1 + \dots + c_s]\,. \tag{7.3}$$

We prove that for every $\varepsilon > 0$ and $M > 0$ there is N such that

$$\mathsf{Pr}(C_{M,N,\varepsilon}) > 0\,, \quad \text{where} \quad C_{M,N,\varepsilon} = [D_N(x,y) < \varepsilon \text{ for all } x, y \text{ with } 0 \le x, y \le M]\,.$$

To show this, let μ be the probability measure on $\mathbb{R}^+ \times \mathbb{R}^+$ governing our SDS, that is, $\mathsf{Pr}[(A_k, B_k) \in U] = \mu(U)$ for any Borel set $U \subset \mathbb{R}^+ \times \mathbb{R}^+$. By our assumptions, there are $(a_1, b_1), (a_2, b_2) \in \mathsf{supp}(\mu)$, such that $0 < a_1 < 1 < a_2$ and $b_1, b_2 > 0$. We choose $\Delta > 1$ such that $a_1\,\Delta < 1 < a_2/\Delta$ and $b_* = \min\{b_1, b_2\}/\Delta > 0$, and we set $b^* = \max\{b_1, b_2\}\,\Delta$.

Let $r, s \in \mathbb{N}$. For $k = r+1, \dots, r+s$, we recursively define indices $i(k) \in \{1, 2\}$ by

$$i(r+1) = 1, \quad i(k+1) = \begin{cases} 1\,, & \text{if } a_{i(r+1)} \cdots a_{i(k)} \ge 1, \\ 2\,, & \text{if } a_{i(r+1)} \cdots a_{i(k)} < 1. \end{cases}$$

Therefore $a_1 \le a_{i(r+1)} \cdots a_{i(k)} \le a_2$ for all $k > r$. We have

$$\mathsf{Pr}[a_2/\Delta^{1/r} \le A_k \le a_2\,\Delta^{1/r} \text{ and } b_* \le B_k \le b^*] > 0\,, \quad k = 1, \dots, r\,, \quad \text{and}$$
$$\mathsf{Pr}[a_{i(k)}/\Delta^{1/s} \le A_k \le a_{i(k)}\,\Delta^{1/s} \text{ and } b_* \le B_k \le b^*] > 0\,, \quad k = r+1, \dots, r+s.$$

Since the (A_k, B_k) are i.i.d., we also get that with positive probability,

$$\begin{aligned} \frac{a_2^k}{\Delta} \le A_{0,k} &\le a_2^k\,\Delta \quad \text{for} \quad k = 1, \cdots, r\,, \\ \frac{a_1}{\Delta} \le A_{r,r+j} &\le a_2\,\Delta \quad \text{for} \quad j = 1, \cdots, s\,, \\ b_* \le B_k &\le b^* \quad \text{for} \quad k = 1, \cdots, r+s\,, \end{aligned}$$

and thus, again with positive probability,

$$\frac{B_k}{A_{0,k}} \le \frac{b^*\,\Delta^2}{a_2} \quad \text{for} \quad k = 1, \cdots, r \quad \text{and}$$
$$\frac{b_*}{a_2^{r+1}\,\Delta^2} \le \underbrace{\frac{B_{r+j}}{A_{0,r+j}}}_{=:\,c_j} \le \frac{b^*\,\Delta^2}{a_1 a_2^r} \quad \text{for} \quad j = 1, \cdots, s\,. \tag{7.4}$$

We now set $M' = b^* \Delta^2/a_2$ and then choose r and s sufficiently large such that

$$\frac{b^*\,\Delta^2}{a_1 a_2^r} < \varepsilon \quad \text{and} \quad s\,\frac{b_*}{a_2^{r+1}\,\Delta^2} \ge M + M'.$$

We set $N = r + s$ and let $C_{N,\varepsilon}$ be the event on which the inequalities (7.4) hold. On $C_{N,\varepsilon}$, we can use (7.3) to get $Z_r^0 \le M'$. Since $D_n(x,y)$ is decreasing in n, we have for $x \in [0\,,\,M]$ that $|Z_r^x - Z_r^0| \le x \le M$ and thus $\xi = Z_r^x \in [0\,,\,M+M']$. Now we can apply (7.3) with c_j as in (7.4) and obtain $\max_j c_j < \varepsilon$ and $c_1 + \dots c_s \ge M + M'$. But for the associated mappings $f_1, \dots, f_s$ according to (7.3), we have $Z_n^x = f_s \circ \cdots \circ f_1(\xi)$. We see that on the event $C_{N,\varepsilon}$, one has $Z_n^x < \varepsilon$ for all $x \in [0\,,\,M]$, whence $D_N(x,y) < \varepsilon$ for all $x, y \in [0\,,\,M]$.

We can use Lemma 6.27 to conclude. $\square$

Combining the last proposition with theorems 6.17 and Theorem 6.32, we obtain the main result of this section.

(7.5) Theorem. *Consider the reflected affine stochastic recursion* (1.1). *Suppose*

(1) *non-degeneracy:* $\mathsf{Pr}[A_n = 1] < 1$ *and* $\mathsf{Pr}[A_n x + B_n = x] < 1$ *for all* $x \in \mathbb{R}$

(2) *moment conditions:* $\mathsf{E}(|\log A_n|^2) < \infty$ *and* $\mathsf{E}\big((\log^+ |B_n|)^{2+\varepsilon}\big) < \infty$ *for some* $\varepsilon > 0$

(3) *centered case:* $\mathsf{E}(\log A_n) = 0$.

Then the SDS has a unique invariant Radon measure ν on $[0\,,\,\infty)$, it is topologicallly recurrent on $\mathsf{L} = \mathsf{supp}(\nu)$. The time shift on the trajectory space $\big([0\,,\,\infty)^{\mathbb{N}_0}, \mathsf{Pr}_\nu\big)$ is ergodic, where $\mathsf{Pr}_\nu = \int_{[0\,,\,\infty)} \mathsf{Pr}_x \, d\nu(x)$ and Pr_x is the probability measure governing the process starting at x.

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